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ABSTRACT

The Sequential Mathematics Sequence provides an integrated course of study for grades 9-11 in New York City. This document presents lesson plans for part two of the second-year course, designed for tenth grade. The curriculum interweaves algebra, geometry, logic, probability, statistics, and trigonometry, with the emphasis in the 60 lessons in this document on trigonometry, geometry, algebra, probability, and number ideas. Each lesson begins with a clearly stated aim presented in question form. Performance objectives, new vocabulary, and a motivational activity follow. A full development is then presented, with suggested pivotal questions. Applications, a summary, and sample homework examples are also included. In each lesson opportunities are infused for helping students to develop critical thinking, problem-solving, and reasoning skills. (MNS)

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SEQUENTIAL MATHEMATICS

Course II

Part 2

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Pilot Edition — 1986

SEQUENTIAL MATHEMATICS

(formerly Integrated Mathematics)

Course II

Part 2

NEW YORK CITY BOARD OF EDUCATION
Office of Curriculum Development and Support
Division of Curriculum and Instruction

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FOREWORD

The Sequential Mathematics Sequence provides New York City students with an integrated course of study for grades nine, ten, and eleven. In a departure from traditional courses of study, lessons interweave algebra, geometry, logic, probability, and mathematical systems. This course extends the integrated approach that begins in kindergarten and continues through grade 8.

In the Sequential Mathematics Sequence, algebra, geometry, and trigonometry remain important components of the curriculum. In addition, other branches of mathematics --such as probability, logic, and mathematical systems -- are introduced and emphasized as important tools for dealing with present and future technological developments.

Each lesson in Course II begins with a clearly stated aim presented in question form. Performance objectives follow; these provide a specific focus for the teacher. The new vocabulary for the lesson is presented. This is followed by a motivational activity. A full development is presented, with suggested pivotal questions. Applications, a summary, and sample homework examples are also included. In each lesson opportunities have been infused for helping students to develop critical thinking, problem-solving, and reasoning skills. These activities will enable students to become more skilled in higher levels of thinking.

The Sequential Mathematics Sequence has been designed to anticipate students' needs for mathematical skills in an increasingly technologically-oriented culture. We anticipate that the Sequential Mathematics Sequence will serve the individual needs of your students and enable them to acquire the mathematics skills they need in the present and for the future.

Charlotte Frank

Charlotte Frank
Executive Director
Division of Curriculum and Instruction

INTRODUCTION

Sequential Mathematics (Course II), Part II is the second year course of the newly implemented New York State curriculum. This publication from the Division of Curriculum and Instruction is a guide for teachers. It attempts to present the material covered in this course in a way that is most comprehensive and useful to the classroom teacher.

Traditionally, in New York State, mathematics in grades nine, ten, and eleven has consisted of algebra, geometry, and intermediate algebra/trigonometry. Students completing the three-year sequence have been exposed to these topics as relatively independent units. Moreover, students discontinuing their study of mathematics after grade eleven have never been exposed to such timely and practical topics as probability, statistics, and formal logical reasoning. Those terminating their high school mathematics program after grade ten have been given even fewer opportunities for exposure to these topics.

The three-year Sequential Mathematics sequence designed by New York State is a move in the direction of fusing these formerly separate topics and introducing those which have been absent from the course of study.

This curriculum guide is based upon the Scope and Sequence for Sequential Mathematics, Course II, Part II. It consists of lesson plans, which should be considered suggestions rather than prescriptions. Each lesson stresses the development of mathematical concepts, with an emphasis on providing opportunities for the development of problem solving skills, reasoning skills and critical thinking. To help students develop these skills, each lesson contains suggested questions to encourage student investigation and thought. The problems presented in the lessons are designed to be relevant to students' experiences. There is an abundance of application and homework problems. These have been provided to help teachers adapt instruction to the individual differences within the classroom.

Though Course II is written with tenth grade students in mind, it is possible that some students may begin the material earlier or later, or complete it in less than or more than one year. It is expected that individual teachers will consider their students' abilities and interests in making judgments concerning time allotments, aspects of material to be stressed, or extra material to be presented.

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Lesson 1

AIM: What is the tangent ratio?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define the tangent of an acute angle.
2. State why the tangent of a given acute angle in a right triangle is constant.
3. Find the values of tan 30°, tan 45° and tan 60° using special triangles.

VOCABULARY: tangent, ratio, special triangles

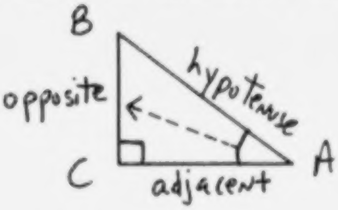
MOTIVATION:

Warm-up: In right triangle ABC, $\measuredangle C$ measures 90°, angle B measures 60° and BC = 10. Find the measure of $\overline{AC}$.
Can we use the Pythagorean Theorem to find the length of $\overline{AC}$?

DEVELOPMENT:

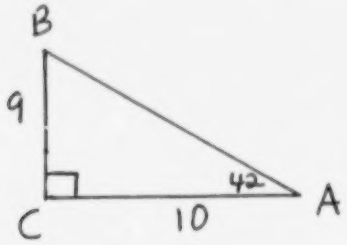
1. Draw a right triangle on the board and indicate or elicit which side is the hypotenuse and which sides are the legs.
2. If available, distribute protractors and rulers and have the students draw right triangles with 60° angles. Elicit that the third angle measures 30°. Have them draw the side between the 90° and 60° angles to various lengths (i.e., 1,2,4,5). Now have them measure the other leg. Elicit the measures of the other sides and form ratios.
   Elicit that the ratio of the length of the side opposite the 60° angle to the length of the side adjacent is approximately 1.7 to 1. State that this ratio is actually $\sqrt{3}:1$.
   Define this ratio as the tangent ratio with the respect to the 60° angle.

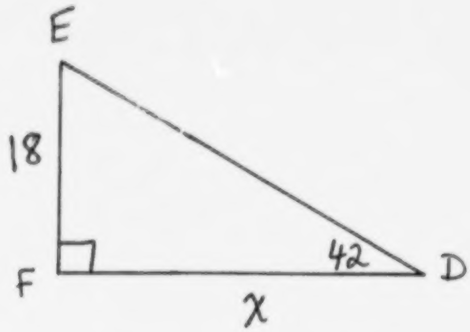
__Definition__: The tangent of an acute angle of a right triangle is the ratio of the length of the side opposite the acute angle to the length of the side adjacent to it. Draw a right triangle



the TAN A = $\frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{\overline{BC}}{\overline{AC}} = \frac{a}{b}$.

3. __Ask__ (Pivotal question): Will there be a ratio for all acute angles? That is, if right triangles ABC and DEF are:





could we find x? Elicit: yes, because the two triangles are similar. Elicit that for every acute angle there is a specific tangent ratio value (i.e., tan 42° $\approx$ .9000).

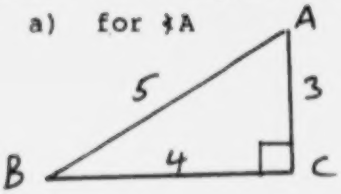
4. Refer back to the 30-60-90 triangle. Ask if the tangent of 60° = $\frac{\sqrt{3}}{1}$, what is the tangent of 30°? Elicit that these are reciprocals, hence the tan 30° = $\frac{1}{\sqrt{3}}$. Rationalizing the denominator, we see the tan 30° = $\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. Tan 30° = $\frac{\sqrt{3}}{3}$.

5. Now draw a right triangle with a 45° angle. Elicit that this is an isosceles right triangle. Why? (Both acute angles measure 45°; if two angles of a triangle are congruent, then it is isosceles.)
   __Ask__: What is the tangent of 45°? (Tan 45° = 1) That is, in $\triangle XYZ$, if $m\angle X = 45$ and $m\angle Z = 90$, if XZ = 10 then what will YZ equal? (10)

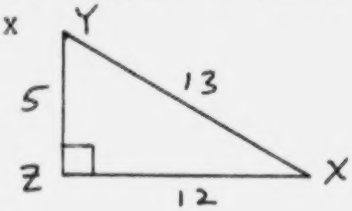
APPLICATIONS:

1. Write the tangent ratio in terms of the lengths of sides.

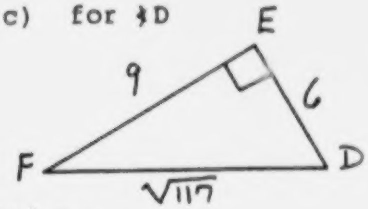
a) for ∢A



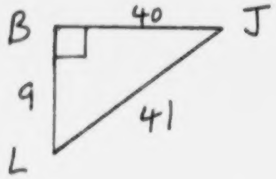
b) for ∢X



c) for ∢D

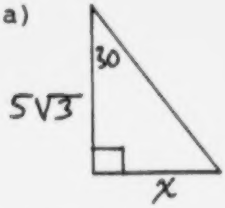


d) for ∢J



2. Find x.

a)



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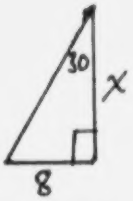

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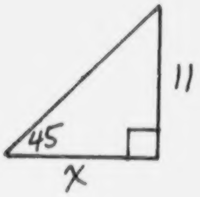


3. Find the tangent of angle A if in △ABC, m∢C = 90, AC = 8 and BC = 12.

4. Find the acute angle of a right triangle if the legs measure 10 and $10\sqrt{3}$.

SUMMARY:

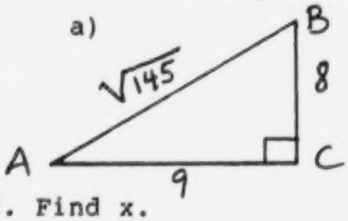
1. What is the tangent ratio? (opp/adj)

2. What is the tangent of 30°? ($\frac{\sqrt{3}}{3}$) 60°? ($\sqrt{3}$) 45°? (1)

HOMEWORK:

1. Write the tangent ratio in terms of sides of ∢A.

a)



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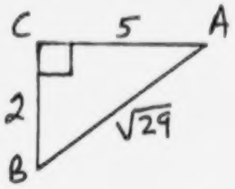

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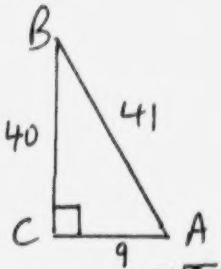
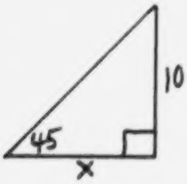


2. Find x.

a)



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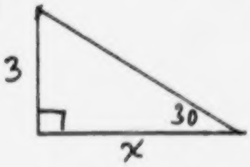

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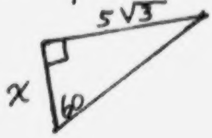


3. Find the tangent of an angle whose opposite side is 10 and whose adjacent side is 16.
4. What is the measure of an acute angle of a right triangle whose legs measure 8 each?

REFERENCES:

Houghton-Mifflin: pp. 336-340
Dressler: pp. 348-352

Lesson 2

AIM: How do we use the tangent ratio to solve a problem?

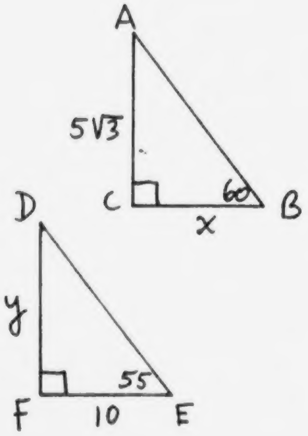
PERFORMANCE OBJECTIVES: The student will be able to...

1. Use a trigometric table to find the tangent of a given acute angle.
2. Use the tangent ratio in problems involving
   - a) angle of elevation
   - b) angle of depression
   - c) angle of elevation of the sum.

VOCABULARY: angle of elevation, angle of depression, trigometric table

MOTIVATION:

Warm-up:



1. Given $\triangle ABC$ with $m\measuredangle B = 60$, find x.

2. Given $\triangle DEF$ with $m\measuredangle E = 55$, find y.

Elicit x = 5 since the tangent ratio is $\frac{\sqrt{3}}{1}$.

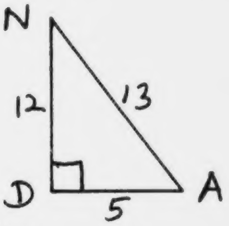
Question: What will we need in order to find y?

DEVELOPMENT:

1. Elicit that we need to know the tangent ratio for 55°. (Distribute tables or use table in the text.) Discuss the fact that there is a specific ratio for every angle. These ratios have been calculated in decimal form and are in the table.
   Ask "What is the tan 55°?" Elicit from the table tangent 55° = 1.4281. Similarly ask the students to find tangent 25°, tangent 74°, ...

2. Elicit that the table could also be used to find angles. Ask which angle has a tangent = 1.9626? (63°) Similarly ask for the angle whose tangent equals .2679, .7813, 11.4301.
   Ask, What do you notice about the value of the tangent for angles that are more than 45°? ...less than 45°?

3. Given right △DAN, find the measure of ∢A.



   Elicit Tangent = $\frac{\text{opp leg}}{\text{adj leg}}$

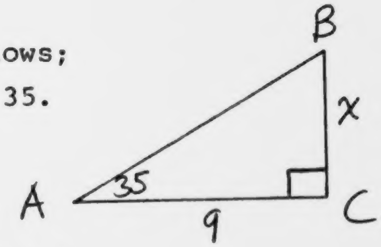
   Ask, What does tangent A equal? $(\frac{12}{5})$

   How can we find the m∢A to the nearest degree? Elicit $\frac{12}{5}$ must be written in decimal form. How do we change $\frac{12}{5}$ into a decimal? Elicit divide the denominator into the numerator. That is, $5\overline{)12.0000}$. Why four decimal places? (Because the table indicates an accuracy of four decimal places.) Therefore, tangent A = 2.4000. What angle has this tangent? Tangent 67°= 2.3559 and tan 68° = 2.4751. Which angle shall we use? Elicit that we must determine which angle is closer to 2.4000.

   $$2.4751 - 2.4000 = 0.075$$
   $$2.4000 - 2.3559 = 0.6441$$

   Therefore 67° is the angle to the nearest degree.

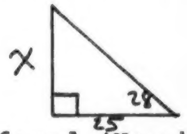
4. Present △ABC as follows; with m∢C = 90, m∢A = 35.



   Ask, How would we find x?

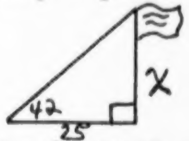
Elicit we use the Tangent ratio, Tan A = $\frac{\text{opp leg}}{\text{adj leg}}$, tan 35° = $\frac{x}{9}$. Ask, What is tan 35°? (.7002) Substituting, we get .7002 = $\frac{x}{9}$. Ask, How do we solve this equation? Elicit if we multiply both sides by 9 we get x = 6.3018.

5. Present the following problem: From a point 25 feet from a building a ladder is placed against the building so that it makes an angle of 28° with the ground. Ask, "How high up on the building does the ladder reach?" Ask, "How would we begin to solve this problem?" (Review various problem solving techniques.)

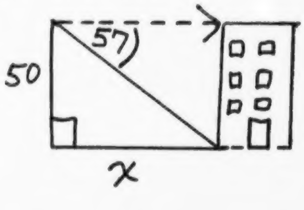


   a) Draw a diagram and label it.
   b) Determine what must be found (How high the ladder reaches).
   c) Define variable - let x equal how high the ladder reaches.
   d) Form an equation. Ask, What other information can we use? Elicit Tan 28° = $\frac{\text{opp leg}}{\text{adj leg}} = \frac{x}{25}$.
   e) Find Tan 25° = .4663.
   f) Solve equation. .4663 = $\frac{x}{25}$. Multiply by 25 and get x = 11.6' to the nearest tenth. (x = 11.6575)

6. Present the following problems.
   a) From a point 25' from the base of the flag pole, the angle of elevation is 42°, find the height of the flag pole. Again following a problem solving technique, we get
   Discuss the meaning of angle of elevation.



   (It is an angle formed by a horizontal ray and a ray of the line of sight when looking up at the object.)
   Elicit the equation to be used, Tan 42° = $\frac{x}{25}$. Have the students complete the solution.
   b) From the top of a scaffold 50 feet high the angle of depression is 57° to the base of the next building. How far apart are the buildings?
   Ask, What do you think is the angle of depression?



(The angle that is formed by a horizontal ray and the line of sight ray when looking down at an object.)

Ask, What is the relationship between the angle of depression and the angle of elevation? Elicit that they are congruent since they are alternate interior angles formed by parallel lines.

Therefore, the equation is Tan $57° = \frac{50}{x}$. $1.5399 = \frac{50}{x}$. Ask, "How do we solve this equation?" Note: First we multiply and then we divide.

$1.5399x = 50$,

$x = \frac{50}{1.5399}$, $x = 32.46 \approx 32$ ft.

APPLICATIONS:

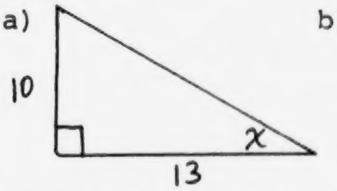
1. Use the table to find the tangent of each angle.
   a) 31° b) 54° c) 3° d) 89°

2. Find the angle whose tangent is
   a) .8693 b) 5.6713 c) .4452

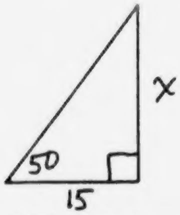
3. From a light house 60 feet high the angle of depression of a boat is 41°. How far from the lighthouse is the boat?

4. In each of the following, find x to the nearest whole number.

a)



b)



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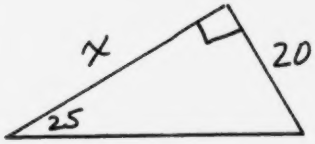


5. A second floor window is 20' high. If a ladder must be placed to make a 30° angle with the ground, how far from the building must the ladder be placed?

6. The angle of elevation of the sun is 40° when a tree casts a shadow of 30 cm. How high is the tree?

SUMMARY:

1. How can we find the length of a side of a right triangle indirectly?

2. What is the angle of elevation? The angle of depression?

3. Under what circumstances would we use the Pythagorean Theorem to find an unknown length of a side of a right triangle?

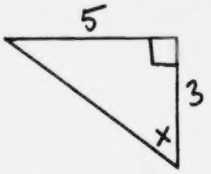
HOMEWORK:

1. Find the tangent of

   a) 11° b) 42° c) 75°

2. Find x to the nearest whole number.

a)



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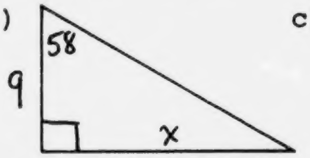

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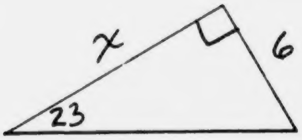


3. How high is a flagpole which casts a shadow of 200 cm when the angle of elevation of the sun is 70°?

4. From the top of a building adjacent to the highway, the angle of depression to the other side is 41°. Find the width of the highway if the building is 90 feet high?

REFERENCES:

Amsco: pp. 353-358
Houghton-Mifflin: pp. 340-346

Lesson 3

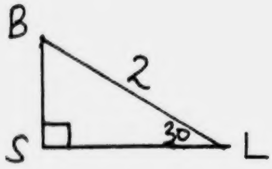
AIM: What are the Sine and Cosine ratios?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define the Sine of an angle.
2. Define the Cosine of an angle.
3. Use special triangles to find the Sine and Cosine of 30°, 45° and 60°.
4. Use a trigometric table to find the Sine and Cosine of a given acute angle.

VOCABULARY: Sine, Cosine

MOTIVATION:



Given $\triangle BLS$ with $m\sphericalangle S = 90$, $m\sphericalangle L = 30$ and $BL = 2$, find BS.
Ask, "Can we use the tangent ratio to solve this problem?"

DEVELOPMENT:

1. Elicit that we can't use the tangent ratio since we don't know the length of either leg. Ask, What do we need? Elicit a ratio involving the hypotenuse. Ask, "What ratio could be used to find BS?" Elicit the ratio $\frac{\text{opposite leg}}{\text{hypotenuse}}$. Define this ratio as the Sine or $\text{Sin} = \frac{\text{opp leg}}{\text{hyp}}$.

   Definition: The SINE of an acute angle of a right triangle is the ratio of the length of the opposite side to the length of the hypotenuse. Therefore, $\text{Sin } L = \frac{x}{2}$, $\text{Sin } 30° = \frac{x}{2}$. Ask, "How can we find the Sin 30°?" (Consult the table.) What is its value? (.5000) Note: This equals $\frac{1}{2}$, therefore we get $BS = 1$.

2. Suppose we wanted to find SL, could we use the Sine ratio? Elicit the need for a third ratio. Define the Cos = $\frac{\text{adj leg}}{\text{hyp}}$. Again we could use the table but what kind of triangle do we have? (30-60-90). Recall that the lengths of the legs are in the ratio of $\sqrt{3}:1$. Therefore, SL = $\sqrt{3}$.

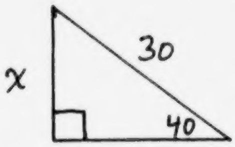
3. Elicit - What is Sine 60°? ($\frac{\sqrt{3}}{2}$) What is COS 60°? ($\frac{1}{2}$)

*Medial Summary: What is the SIN and COS of the acute angles in a 30°-60°-90° triangle?

SIN 30° = $\frac{1}{2}$	COS 30° = $\frac{\sqrt{3}}{2}$
SIN 60° = $\frac{\sqrt{3}}{2}$	COS 60° = $\frac{1}{2}$

4. What other special right triangle is there? (Right Isosceles) Elicit SIN 45° = $\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. What will COS 45° be? (same, $\frac{\sqrt{2}}{2}$)

5. Find x to the nearest whole number:



What ratio should we use?

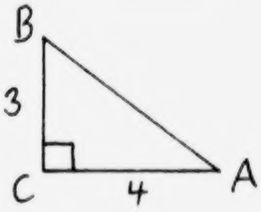
Elicit SIN 40 = $\frac{x}{30}$. We must use the trig table. Sin 40° = .6428, $.6428 = \frac{x}{30}$, $x = 19.28 \approx 19$.

Could we use COS? Yes, but we need the other acute angle. Note: COS 50 = $\frac{x}{30}$, COS 50° = .6428. Hence same solution.

Find SIN 15°, COS 82°, SIN 65°, COS 12°.

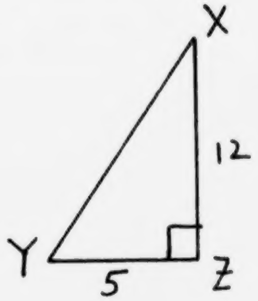
6. Given △ABC.

What is SIN A?



Elicit - we need the length of the hypotenuse.

How do we find the hypotenuse? Use the Pythagorean Theorem. We get AB = 5 and Sin A = $\frac{3}{5}$. What is Cos A? ($\frac{4}{5}$)



Find Cos X if

Note: YX = 13

cos x = $\frac{12}{13}$, sin x = $\frac{5}{13}$

What is cos Y? ($\frac{5}{13}$) Sin y? ($\frac{12}{13}$)

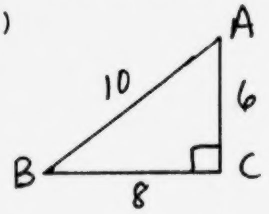
7. What do we notice about sine and cosine ratios? Elicit the Sine of an angle = the Cosine of its complement. That is

Sin 20° = Cos 70°   Sin 60° = Cos 30°
Sin 30° = Cos 60°   etc.

APPLICATIONS:

1. Find the Sine and Cosine of each acute angle.

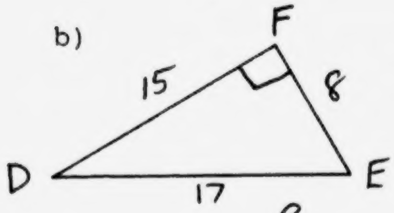
a)



Ans: Sin A = $\frac{8}{10}$, Sin B = $\frac{6}{10}$

Cos A = $\frac{6}{10}$, Cos B = $\frac{8}{10}$

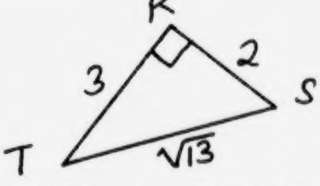
b)



Ans: Sin D = $\frac{8}{17}$, Sin E = $\frac{15}{17}$

Cos D = $\frac{15}{17}$, Cos E = $\frac{8}{17}$

c)



Ans: Sin T = $\frac{2}{\sqrt{13}} \cdot \frac{\sqrt{13}}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$, etc.

2. Use the table to find the value of each of the following.
   - a) Sin 5°
   - b) Sin 85°
   - c) Cos 74°
   - d) Sin 37°
   - e) Cos 2°
   - f) Sin 60°

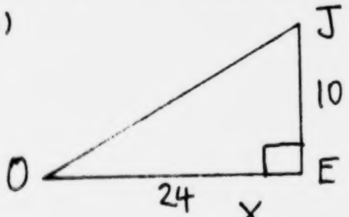
3. Find each angle to the nearest degree.
   - a) Sin A = 0.2079
   - d) Cos M = 0.3000

b) Cos H = 0.3907 e) Sin T = 0.5555

c) Sin B = 0.2450 f) Cos X = .5000

4. Find the Sine and cosine of the acute angles.

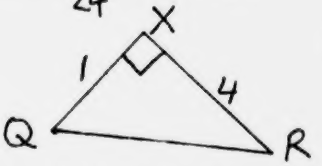
a)



Ans: JO = 26

$\text{Sin } O = \frac{10}{26}$, etc.

b)

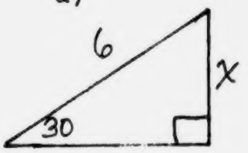


Ans: $QR = \sqrt{17}$

$\text{Sin } Q = \frac{4}{\sqrt{17}} = \frac{4\sqrt{17}}{17}$

5. Find x in each of the following.

a)



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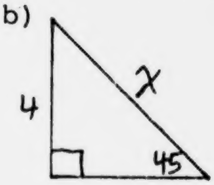

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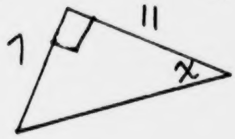

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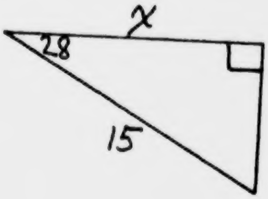


SUMMARY:

1. What are the Sine and Cosine ratios of an acute angle of a right triangle?
2. If the Sine of an angle is equal to the Cosine of 4 times the angle, what is the measure of the angle?

HOMEWORK:

1. Find the value of each of the following.

a) Sin 32° c) Cos 11°

b) Cos 60° d) Sin 85°

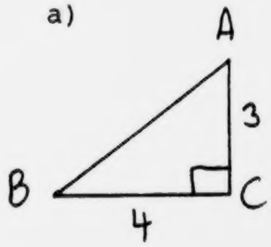
2. Find each angle to the nearest degree.

a) Sin A = .5000 c) Cos N = .2500

b) Cos B = .7071 d) Sin M = .1000

3. Find the Sin and Cos of angle A.

a)



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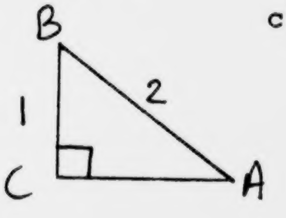

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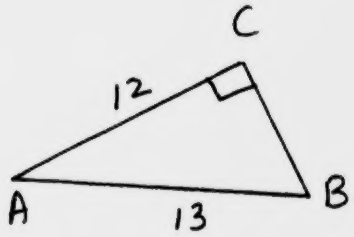
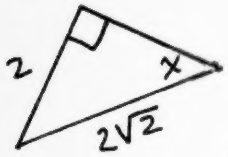


4. Find x.

a)



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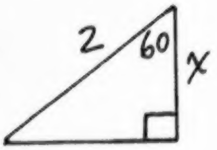

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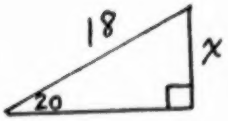

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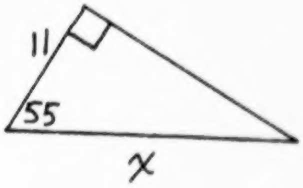


REFERENCES:

Amsco: pp. 358-363

Houghton-Mifflin: pp. 336-340

Lesson 4

AIM: How do we use the Sine and Cosine ratios to solve problems?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Solve a verbal problem using Sine and Cosine ratios.
2. Use the Sine or Cosine ratio to find the measure of an angle to the nearest degree.

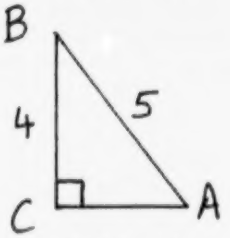
VOCABULARY: Sine, Cosine

MOTIVATION:

In △ABC, BC = 4, AB = 5 and m∢C = 90. Find ∢A to the nearest degree.
How can we find the m∢A?

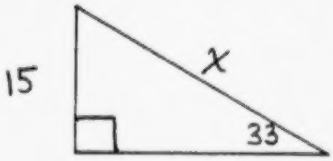
DEVELOPMENT:

1.



Elicit the Sin A = $\frac{4}{5}$.
Changing $\frac{4}{5}$ to .8000, we use the table to find m∢A. (53°)

2. Present the following problem.
How long must an escalator be if it must be constructed at an angle of 33° with the floor and carry people to a floor 15 feet above the preceding floor.



Elicit - draw diagram.
Ask - What ratio should we use? (Sin)
Sin 33° = $\frac{15}{x}$, Sin 33° = .5446
Therefore, .5446 = $\frac{15}{x}$, .5446 x = 15, x = $\frac{15}{.5446}$,
x = 27.54 feet.

3. A ladder 40 feet in length is leaning against a building at an angle of 65° with the ground. How far from the building is the foot of the ladder? (Express answer to the nearest foot)

Again, lead the students through the procedure, eliciting all of the steps.

$\text{Cos } 65^\circ = \frac{x}{40}$, $\text{Cos } 65^\circ = 0.44226$,

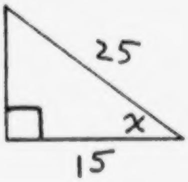
$0.4226 = \frac{x}{40}$ and $x = 40(.4226) = 16.9$, $x \approx 17$ feet to the nearest foot.

4. Present the following problem.

   Find to the nearest degree the angle a pole 25 meters long makes with the ground when it is leaning against a wall 15 meters away from the foot of the pole.

   Have the students set up the problem. Ask, "What do we have to find in this problem?" Elicit the remaining steps.

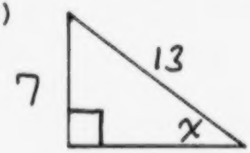
   $\text{Cos } x = \frac{15}{25}$, $\text{Cos } x = .6000$, $x = 53^\circ$



APPLICATIONS:

1. Find the angle to the nearest degree.

a)



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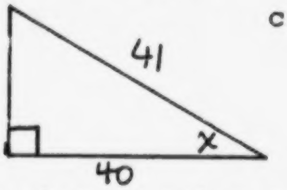

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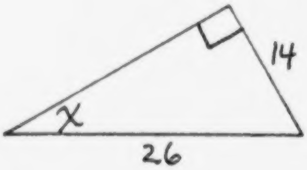


2. A guy wire is sturdiest when it makes an angle of 35° with the ground. How long must a guy wire be to hold up a pole 20 feet tall?

3. A plank of wood 4 meters long makes an angle of 50° with the ground when leaning against a tree. How high on the tree does the plank reach?

4. A ladder 30 feet in length makes an angle of 40° with the ground when leaning against a building. How far from the building is the foot of the ladder?

SUMMARY:

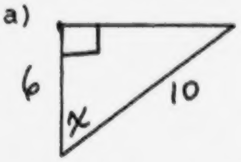
What should we do at the outset to solve problems?

1) Draw a diagram.
2) Label the diagram with the given information and assign a variable to the unknown quantity.
3) Determine which trigonometric ratio is to be used.

HOMEWORK:

1. Find the angle to the nearest degree.

a)



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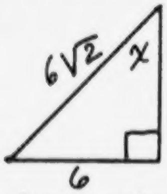

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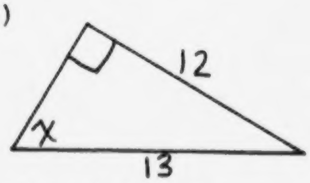


2. A kite string 300 feet long makes an angle with the ground of 65°. How high is the kite flying?

3. A board makes an angle with the ground of 52°. If the board is 10 feet from a building that it is leaning against, how long is the board?

4. If a guy wire makes an angle of 37° with the ground, how long must it be to hold up a 15 foot pole.

REFERENCES:

Amsco: pp. 363-371

Houghton-Mifflin: pp. 340-346

Lesson 5

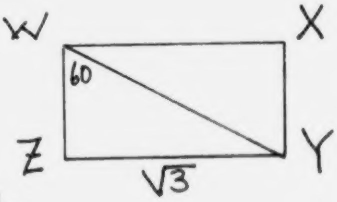
AIM: How do we use trigometric ratios to find lengths or angles in a geometric figure?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Find the missing lengths (diagonals and sides) of a triangle and a quadrilateral.
2. Find missing lengths (apothem, radius, side) of a regular polygon.
3. Find missing angle measures of a polygon (to the nearest degree).

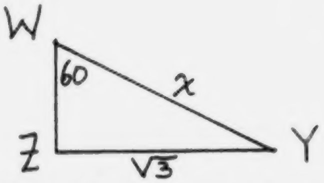
VOCABULARY: apothem, radius, polygon

MOTIVATION:

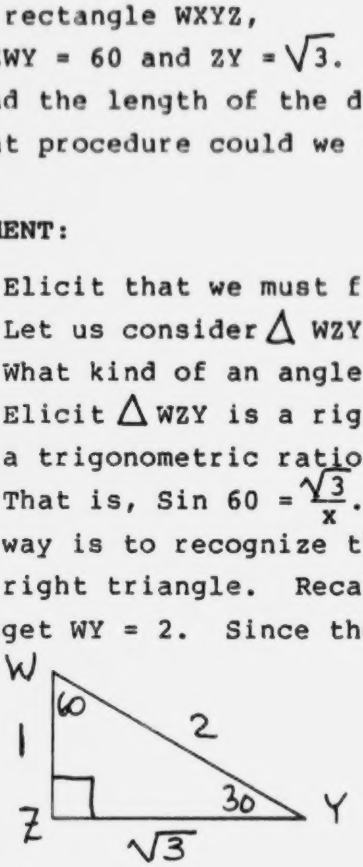


In rectangle WXYZ,
m∡ZWY = 60 and ZY = $\sqrt{3}$.
Find the length of the diagonal.
What procedure could we use to solve this problem?

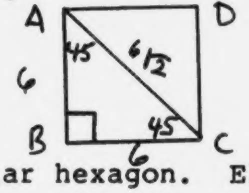
DEVELOPMENT:



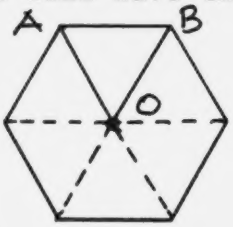
1. Elicit that we must find WY. Let us consider △ WZY. Ask, What kind of an angle is ∡Z? Elicit △WZY is a right triangle and therefore we can use a trigonometric ratio to find WY. That is, Sin 60 = $\frac{\sqrt{3}}{x}$. We can do this in two ways. One way is to recognize that WYZ is a special 30°-60°-90° right triangle. Recalling the special relationships, we get WY = 2. Since the triangle sides measure 1, $\sqrt{3}$, 2 as shown. This gives us an exact measure. Using the table we get .8660 = $\frac{3}{x}$. x(.8660) = 3 and x = $\frac{1.732}{.8660}$ = 2.



2. Find the diagonal of a square whose side is 6. Since the adjacent sides are congruent, △ABC is an isosceles right triangle. Therefore m∢BAC = 45, so we have a special 45°-45°-90° right triangle and AC = $6\sqrt{2}$.

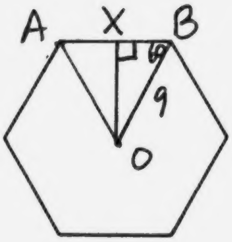


3. Let us consider a regular hexagon. Elicit the definition and draw it. This and other regular polygons have special interior segments. Consider point O, which would be considered the center of the regular hexagon. If we draw $\overline{OA}$, $\overline{OB}$, etc. we will have the radii of the polygon.



What kind of triangle is △AOB? Elicit it is equilateral. This can be seen by drawing a circle around the hexagon (circumscribing a circle). ∢AOB would be a central angle and since all such angles would be congruent, then m∢AOB = 60. Since $\overline{AO}$ and $\overline{BO}$ are radii, we have an isosceles triangle with a 60° vertex angle and therefore △AOB is equilateral. If AB = 9, what is the measure of any radius of the hexagon? (9) Will this be true of all regular polygons? (NO)

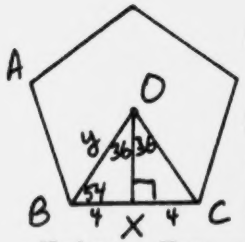
4. Consider $\overline{OX}$ where $\overline{OX} \perp \overline{AB}$. $\overline{OX}$ is called the apothem of the regular polygon.
Elicit the definition: an apothem of a polygon is the perpendicular segment drawn from the center to a side.



How can we find the length of $\overline{OX}$? Consider △XOB. It is a 30°-60°-90° right triangle with hypotenuse OB = 9.

Therefore, the apothem equals $4.5\sqrt{3}$ from a special right triangle relationship. Could we use trigonometry to find this? Yes, $m\angle XOB = 60$. Therefore the $\sin B = \frac{x}{9}$, $\sin 60^\circ = \frac{x}{9}$. $.8660 = \frac{x}{9}$ and $x = 9(.8660) = 7.8 \approx 8$.

5. Could we find the lengths of the radius and apothem of any regular polygon? (Yes, if we know the length of a side.) Find the lengths of the radius and apothem of a regular pentagon whose side has measure 8.



Since the radii $\overline{OB}$ and $\overline{OC}$ are congruent, $\triangle OBC$ is isosceles and, therefore, if $BC = 8$, $XB = 4$. What is the measure of central $\angle BOC$? $(\frac{360}{5} = 72)$ Therefore, $m\angle BOX = 36$ and $m\angle XBO = 54$.

Note: You could also find $m\angle ABC = \frac{(5-2)180}{5} = 108$ and thus $m\angle XBO = \frac{1}{2}(108) = 54$.

Using $\cos B = \frac{4}{y}$, we get $\cos 54^\circ = \frac{4}{y}$, $0.5878 = \frac{4}{y}$, $y(0.5878) = 4$, $y = \frac{4}{0.5878}$ and $y = 6.6$.

Therefore, the radius = 7.

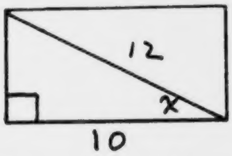
How could we find the apothem? (use the Tangent ratio.)

$\tan 54^\circ = \frac{a}{4}$, $1.3674 = \frac{a}{4}$ and $a = 5.47$ or $a \approx 5.5$

6. If the length of a side of a rectangle is 10 and the length of a diagonal is 12, what is the measure to the nearest degree of the angle the side makes with the diagonal.

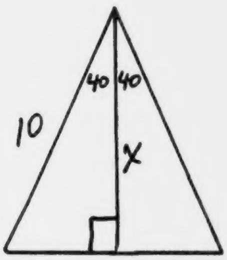
$\cos X = \frac{10}{12} = \frac{5}{6}$

$\cos X = .8333$ and $x = 34^\circ$.



7. What is the length of an altitude of an isosceles triangle with a leg having a measure of 10 and whose vertex angle measures 80.

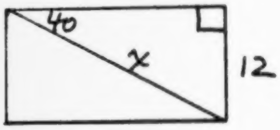
$\cos 40^\circ = \frac{x}{10}$



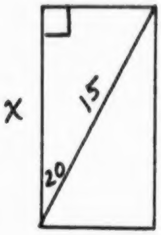
APPLICATIONS:

1. Find x.

a)



b)



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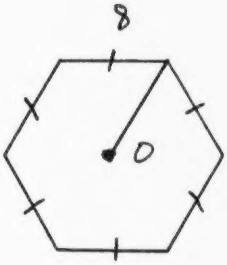


2. Find the length of an altitude of an equilateral triangle whose side measures 10.

3. Find the length of an apothem of each of the following regular polygons.
    a) pentagon (length of side = 10)
    b) octagon (length of side = 12)
    c) decagon (length of side = 5)

4. Find the angle formed by a side of a rectangle and the diagonal if the side is 6 cm in length and the diagonal is 8 cm in length.

5. Find the length of a leg of an isosceles triangle whose altitude to the base measures 8 and whose base angles measure 50° each.

SUMMARY:

What information do we need to be given in order to be able to calculate the length of:

1. The diagonal of a rectangle?

2. The altitude of an isosceles triangle?

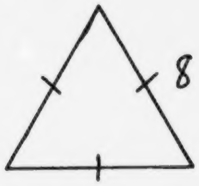
3. The apothem of a regular polygon?

(Note: In each case elicit alternate methods of solution as the data given are changed.)

HOMEWORK:

1. Find the length of the altitude for each of the following.

a)



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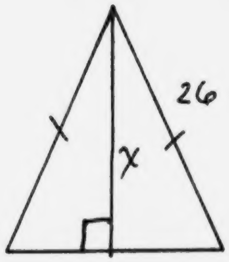


2. Find the lengths of the apothem and radius of a seven-sided regular polygon whose side measures 15.
3. What is the measure of an apothem of a regular hexagon whose side measures 30 cm?
4. Find the length of a diagonal which makes a 35° angle with a side of a rectangle that is 40 cm long?

REFERENCES:

Amsco: pp. 256-258
Houghton-Mifflin: pp. 163-166

Lesson 6

AIM: How do we use trigometric ratios to find the area of a polygon?

PERFORMANCE OBJECTIVES: The student will be able to...

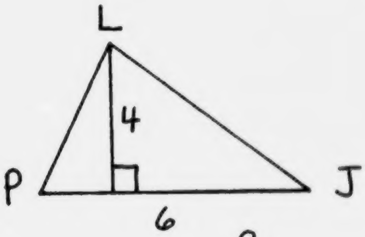
1. Find the area of a triangle.
2. Find the area of a special quadrilateral.
3. Find the area of a regular polygon.

VOCABULARY: trigometric, quadrilateral

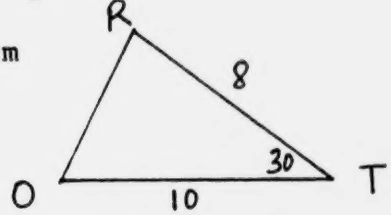
MOTIVATION:

What is the area of

a) $\triangle$PLJ



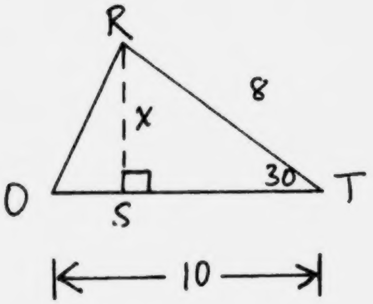
b) $\triangle$ORT
if $m\angle T = 30$, OT = 10 cm
and RT = 8 cm



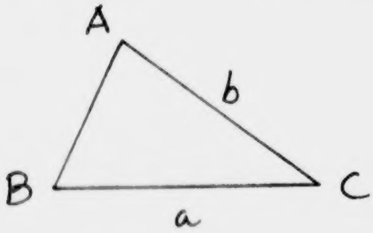
DEVELOPMENT:

1. Elicit that the area of $\triangle$PLJ can be found by using the formula $A = \frac{1}{2}bh$. Review the meaning of b and h as the base the times height to that base. Substituting:
$$A = \frac{1}{2}bh, \quad A = \frac{1}{2}\cdot 6\cdot 4 = 3\cdot 6 = 12 \text{ or } \frac{1}{2}\cdot 24 = 12$$

2. Can we use this formula for the area of the second triangle? Elicit that the height is not known. Ask, "Can we find the height?" Draw the diagram.

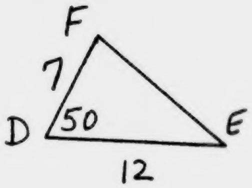


Ask, "What kind of angle does $\overline{RS}$ make with $\overline{OT}$?" Elicit the height is perpendicular to the base. Therefore, $\triangle$RST is a right triangle. Can we find RS? Let x = RS. Elicit, $\sin T = \frac{RS}{RT}$, $\sin 30 = \frac{x}{8}$, $.5 = \frac{x}{8}$. Therefore, x = 4. Now can we find the area? $A = \frac{1}{2}\cdot 4\cdot 8 = 16$. Could we always do this? (yes) Let's generalize. $\sin T = \frac{h}{RT}$, therefore, $h = \sin T \cdot RT$, substituting we get $A = \frac{1}{2}bh = \frac{1}{2}b\cdot \sin T\cdot RT = \frac{1}{2}(b)(RT)\sin T$. Where must the angle be in relation to the sides? (included) Therefore, given,



the area = $\frac{1}{2}ab\cdot\sin C$.

3. Given $\triangle$ DEF with m∢D = 50, ED = 12 and FD = 7, find the area.



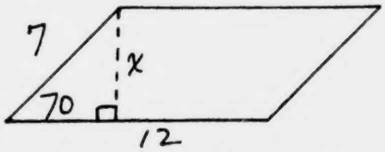
$A = \frac{1}{2}ab\cdot\sin C$ $\quad A = \frac{1}{2}\cdot 12\cdot 7\cdot\sin 50°$

$\sin 50° = .7660$ $\quad A = \frac{1}{2}\cdot 12\cdot 7(.7660)$

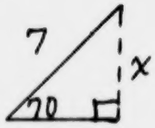
$A = 42(.7660) = 32.17$

A = 32 sq. units

4. Find the area of a parallelogram with sides of 7 and 12 and one angle of 70°.



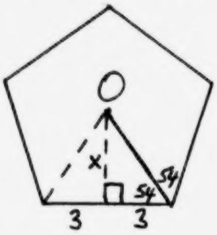
What do we need to know in order to find the area? Elicit A p = bh. Therefore, we need the height. Similar to the method used for triangles, we use trigonometry. $\sin 70° = \frac{x}{7}$, $.9397 = \frac{x}{7}$ and x = 6.58. Therefore, Area = $12\cdot 6.58 = 78.9 \approx 79$.



How could we find this in another way? Elicit an analogous formula, A p = absinC, where ∢C is the included angle.

5. Let us consider a regular octagon with the length of a side = 10. How can we find its area? Elicit that to do this we could divide the octagon into triangles. Ask, "How many triangles are there?" (8), "What would we need to find the area of one of the triangles?" Elicit, the height of the triangle which is the apothem of the polygon. Let x=apothem. Review each angle measures $\frac{(8-2)180}{8} = \frac{1080}{8} = 135$. Therefore TAN 62.5° = $\frac{x}{5}$. Let's use 63°, we get $1.9626 = \frac{x}{5}$ and $x = 9.8$. Therefore the area of the triangle is $\frac{1}{2}\cdot 10\cdot(9.8) = (4.9)\cdot 10 = 49$ square units. What do we do to get the area of the octagon? (multiply by 8) 49 x 8 = 392 sq. units.
Is there a shorter way? Consider ($\frac{1}{2}$apothem·side)·8. That is, the area of each triangle times the number of triangles (this is equal to the number of sides). If we use the associative property, we get $\frac{1}{2}a(S\cdot 8)$. What have we found when we multiply the length of a side by the number of sides? Elicit this is the perimeter. Therefore $\frac{1}{2}$·apothem·perimeter = area of an octagon. Will this be true for any other polygons? (yes - all regular polygons) Hence, we have the formula that the area of a regular polygon is $\frac{1}{2}ap$.

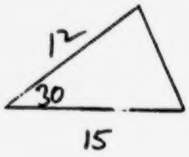
6. Find the area of a regular pentagon whose side measures 6.
$A = \frac{1}{2}ap$, find a.
Tan 54° = $\frac{a}{3}$, $1.3764 = \frac{x}{3}$ and
a = 4.12 4. What is p? (5·6 = 30)
Therefore, $a = \frac{1}{2}\cdot 4\cdot 6 = 12$ sq. units.



APPLICATIONS:

1. Find the area of the given triangles.

a)



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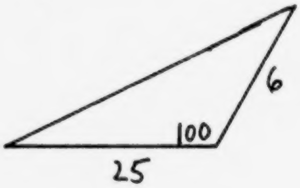
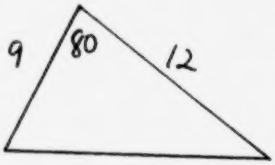
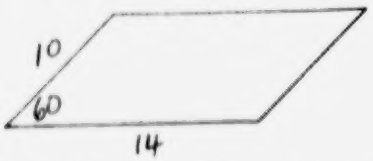


c)



2. Find the area of the given parallelograms.

a)



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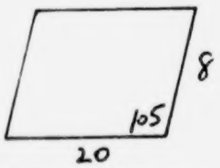

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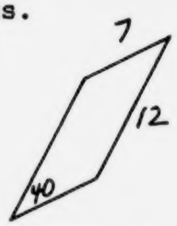


3. Find the area of a regular pentagon the length of whose side is 10.

4. Find the area of a regular octagon the length of whose side is 12.

5. Find the area of a regular decagon the length of whose side is 8.

6. Find the area of a parallelogram whose side measures 8, and whose diagonal measures 13 when this side and the diagonal form a 35° angle. (Hint: Find the area of the triangle and double.)

SUMMARY:

1. How do we find the area of a triangle given the length of two sides and the measure of the included angle?

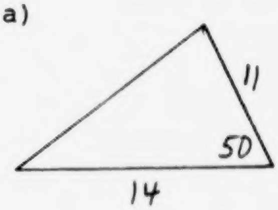
2. What is the formula for the area of a regular polygon?

3. Could we apply the formula $A=\frac{1}{2}ap$ in finding the area of a right triangle? Why?

HOMEWORK:

1. Find the area of the given triangles.

a)



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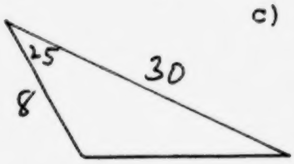

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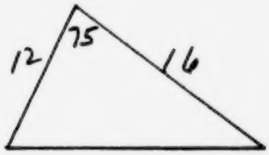
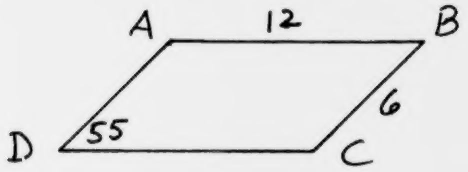


2. Find the area of the regular polygons indicated if their sides measure as follows.
   a) hexagon with side = 9
   b) septagon (7-sided) with side = 10
   c) octagon with side = 15

3. Find the area of a rhombus whose sides each measure 9 and an angle measuring 110.

4. Find the area of parallelogram ABCD.



5. Find the area of a square whose apothem has a length of $5\sqrt{2}$.

6. The area of a regular hexagon is $72\sqrt{3}$. If the length of an apothem is $8\sqrt{3}$, what is the length of a side?

REFERENCES:

Amsco: pp. 256-258
Houghton-Mifflin: pp. 163-166

Lesson 7

AIM: What is a rectangular coordinate system?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Explain the meaning of the terms coordinate geometry and analytic geometry.
2. Define or describe each of the following: coordinate axes, origin, quotient, abscissa, ordinate, coordinates of a point.
3. Draw and label coordinate axes and number the quadrants.
4. Graph ordered pairs of real numbers and state the coordinates of any point in the plane.
5. Explain the meaning of rectangular coordinate system.

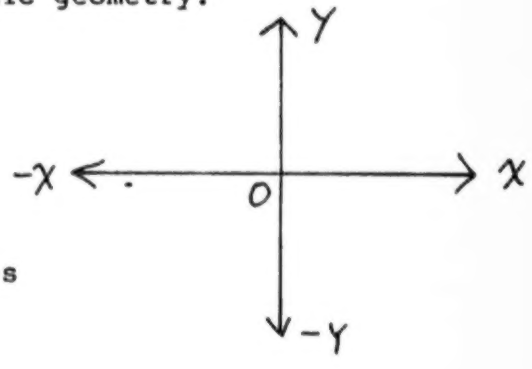
VOCABULARY: coordinate geometry, analytic geometry, coordinate axes, origin, abscissa, quadrant, coordinates

MOTIVATION:

If a boat is at sea and in danger, how does it give its location?

DEVELOPMENT:

1. Elicit that coordinates would be given such as 7 mi. North, 10 mi. South of meridian lines. Review that there is a branch of mathematics that uses coordinates called coordinate geometry or analytic geometry.
2. Draw a set of axes and review the following:
   a) axes, x - horizontal, y - vertical
   b) origin - the point of intersection of the axes



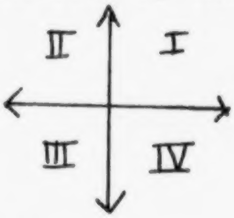
c) points - i.e., (2,1) Elicit: 2 is the x value, 1 is the y value.

d) coordinates - an ordered pair of numbers that describes the location of a point in the coordinate plane.

e) abscissa: the x-coordinate value, the abscissa of (2,1) is 2.

ordinate: the y-coordinate value, the ordinate of (2,1) is 1.

3. Ask, "How many parts is the plane divided into by the axes?" Elicit, 4 parts.
These are called quadrants.
Label and draw the quadrants.



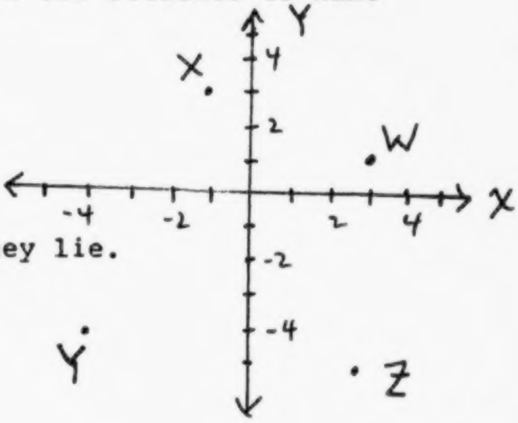
4. Review plotting points. Have the students plot
   - a) A(3,4)
   - b) B(-5,2)
   - c) C(-3,-4)
   - d) D(7,-2)

Elicit - In which quadrants are points A, B, C, D located?

5. Present points W, X, Y, Z and ask the students to name the coordinates.

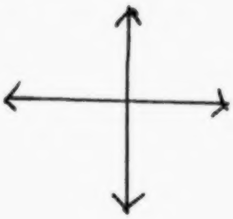
Ans: W(3,1)
X(-1,3)
Y(-4,-4)
Z(3,-5)

Elicit the quadrants in which they lie.



APPLICATIONS:

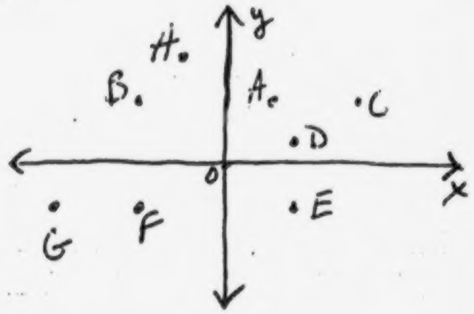
1. Using this graph, label the quadrants.
Answer:



2. Name the abscissa and ordinate of each of the following.
   - a) (6,4)
   - b) (-3,5)
   - c) (-7,-2)
   - d) (a,b)

3. Plot the following points:
   - a) (2,3)
   - b) (3,2)
   - c) (8,1)
   - d) (5,-3)
   - e) (-6,2)
   - f) (-6,-4)
   - g) (1,-5)
   - h) (-3,4)

4. Name the coordinates of the points A-H on the graph.



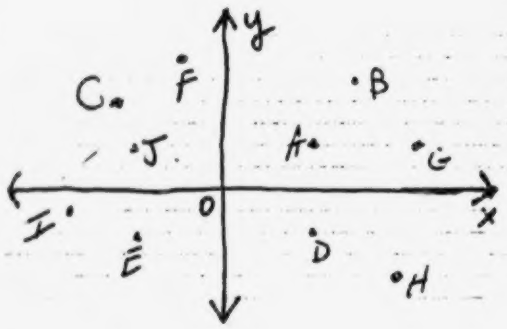
5. Without plotting, name the quadrant in which each point lies.
   - a) (1,1)
   - b) (1,-1)
   - c) (-1,-3)
   - d) (-5,2)
   - e) (-1,4)

SUMMARY:

1. What is the coordinate plane?
2. In which quadrant does (4,-3) lie?
3. What is the ordinate of (4,-3)?
4. In which quadrants may a point lie if its abscissa and ordinate have the same sign? Opposite signs?

HOMEWORK:

1. Name the points on the graph.



2. Plot each of the following on a graph.
   a) (3,4); b) (5,1); c) (4,-2); d) (7,-4); e) (-5,-3);
   f) (-2,4); g) (-3,-3); h) (-1,8); i) (-5,-9)

3. Name the quadrant in which each point lies.
   a) (2,5); b) (-6,-1); c) (3,-2); d) (-5,4); e) (6,1)

Lesson 8

AIM: How do we find the distance between two points in the plane?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Develop and apply the formula for the distance between two points having the same ordinate or abscissa.
2. Develop the formula for the distance between any two points in the plane, using the Pythagorean Theorem.
3. Use the distance formula to find the length of a line segment, sides of triangles and quadrilaterals, and radii of circles.
4. Use the distance formula to show specific relationships for given geometric figures.

VOCABULARY: distance formula, radii

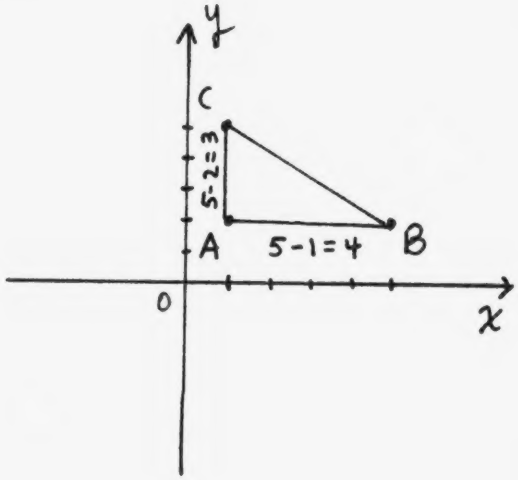
MOTIVATION:

Plot the points and connect them to form $\triangle ABC$.

A) (1,2) B) (5,2) C) (1,5)

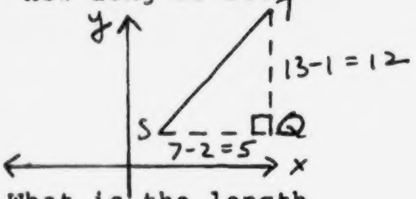
What are the lengths of the sides?

DEVELOPMENT:



1. Send a student to the board to plot A,B,C.
   Elicit - AB = 4, AC = 3.
   How did we find these lengths? (count the boxes)
2. How would you find BC?
   Use Pythagorean Theorem.
   $BC^2 = 3^2 + 4^2$
   $BC^2 = 9 + 16$
   $BC^2 = 25$
   $BC = 5$

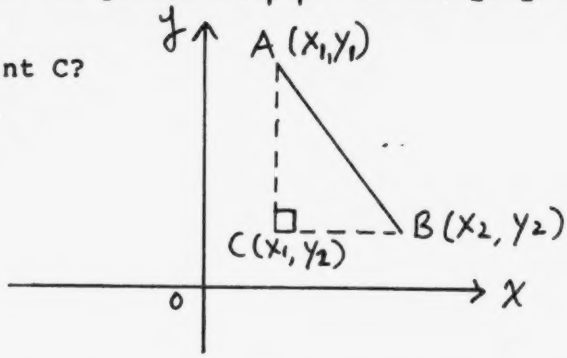
Consider the points $S(2,1)$ and $T(7,13)$ How long is $\overline{ST}$? How could we find its length? Elicit: draw two segments $\overline{SQ}$ and $\overline{TQ}$ such that $\overline{TQ} \perp \overline{SQ}$ and $\overline{SQ}$ is parallel to the X-axis.



What are the coordinates of Q? (7,1) What is the length of $\overline{SQ}$? (5) of $\overline{TQ}$? (12) Again using the Pythagorean Theorem we get $(\overline{ST})^2 = 5^2 + 12^2 = 25 + 144 = 169$. $ST = 13$

Ask, "Do we want to have to do this every time we wish to find the length of a segment?" Elicit we should try to find a formula to use.

Ask, "Is there a relationship between the coordinates of the points S and T and the lengths of the segments SQ, and TQ?" Try to let the students discover that $7-2 = 5$ and $13-1 = 12$. Go back to the first problem and see if that also happened. (Note: $5-2 =3$, $5-1 =4$) Can we generalize this for two points $A(X_1,Y_1,)$ and $B(X_2,Y_2)$ What would be the coordinates of Point C? Elicit, $C(X_1,Y_2)$.



Therefore, what is the length of $\overline{AC}$? (Y_1-Y_2) What is the length of $\overline{BC}$? (X_2-X_1) Using the Pythagorean Theorem we get $(\overline{AB})^2 = (Y_1-Y_2)^2 + (X_2-X_1)^2$ (Note D(AB) is distance of AB) and $D(AB) = \sqrt{(Y_1-Y_2)^2 + (X_2-X_1)^2}$ by taking the square root of both sides.

3. Find the length of LM if $L(8,11)$ and $M(2,3)$

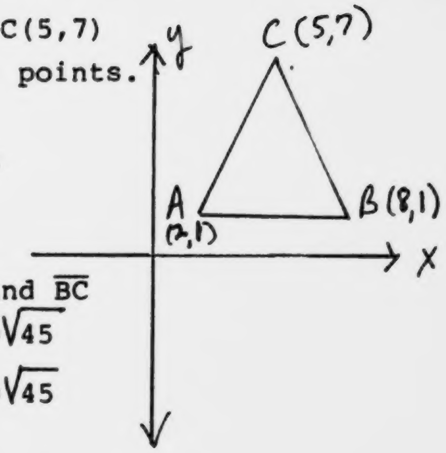
$D = \sqrt{(Y_1-Y_2)^2 + (X_1-X_2)^2}$ Let L be (X,Y)

$D= \sqrt{(11-3)^2 + (8-2)^2} = \sqrt{8^2 + 6^2} = \sqrt{64 + 36} = \sqrt{100} = 10$

Find the distance between $J(-2,+5)$ and $K(2,{}^{-}7)$ again we use the formula.

$$D = \sqrt{(Y_1-Y_2)^2 + (X_1Y_2)^2} = \sqrt{(-7-5)^2 + (2-{}^{-}3)^2}$$
$$= \sqrt{({}^{-}12)^2 + (+5)^2} = \sqrt{144 + 25}$$
$$= \sqrt{169} = 13$$

4. Plot the points A(2,1), B(8,1) and C(5,7) What kind of $\triangle$ is $\triangle$ABC? Plot the points.



Ask: What kind of triangle does it appear to be? (Isosceles) How can we prove it? Use the distance formula to find the lengths of $\overline{AC}$ and $\overline{BC}$

D $\overline{AC} = \sqrt{(7-1)^2 + (5-2)^2} = \sqrt{36 + 9} = \sqrt{45}$

D $\overline{BC} = \sqrt{(7-1)^2 + (5-8)^2} = \sqrt{36 + 9} = \sqrt{45}$

Therefore $\triangle$ ABC is isosceles.

5. Ask: Could we use the distance formula to show a quadrilateral is a parallelogram? Elicit we can by showing that opposite sides are equal in length. Plot A(-4,2), B(3,2) C(5,6), D(2,6). By counting we see AB = 7 and DC = 7. Using the formula we see

D $\overline{AD} = \sqrt{(6-2)^2 + ({}^{-}2-{}^{-}4)^2} = \sqrt{16+4} = \sqrt{20}$

D($\overline{BC}$) $= \sqrt{(6-2)^2 + (5-3)^2} = \sqrt{16+4} = \sqrt{20}$

Hence ABCD is a parallelogram since AB = DC and AD = BC.

APPLICATIONS:

1. Find the distance between each of the following pairs of points.
   a) (4,6) and (8,6)
   b) (7,2) and (7,11)
   c) (-4,3) and (6,3)
   d) (5,2) and (2,6)
   e) (-2,-8) and (3,4)
   f) (2,1) and (3,3)
2. Find the lengths of each segment whose endpoints are given.
   a) (1,1) and (7,-7)
   b) (-4,5) and (4,$^-$10)
   c) (4,3) and (0,0)
   d) (6,4) and 2,3)
3. What kind of triangle is determined by each of the following sets of 3 points.
   a) (4,2), (6,5) and (2,$^-$1)
   b) (3,4), (8,4) and (7,7)
4. Which of the following represent the vertices of a right triangle?
   a) (2,1), (2,5) and (6,1)
   b) (2,-1), (3,2) and 5,-3)
   c) (2,1), (9,9) and (12,-8)

5. What type of quadrilateral is formed by connecting each of the following.
   - a) (3,4), (3,8), (9,4) and (9,8)
   - b) (0,1), (3,5), (9,5) and (6,1)

SUMMARY:

1. How do we find the distance between 2 points?
2. How can we determine the type of triangle or quadrilateral formed given the coordinates of its vertices?

HOMEWORK:

1. Find the distance between each pair of points.
   - a) (5,2) and $(5,{}^{-}4)$
   - b) (6,3) and (2,3)
   - c) (1,1) and (5,4)
   - d) $({}^{-}2,4)$ and $({}^{-}5,{}^{-}8)$
   - e) $({}^{-}2,{}^{-}6)$ and (4,2)
2. Determine the type of triangle formed by the three vertices.
   - a) (0,0), (0,6), (4,0)
   - b) (2,1), (2,6), (6,3)
   - c) (4,1), $(5,{}^{-}2)$ and $(4,{}^{-}3)$
3. What kind of quadrilateral is formed by connecting each of the following?
   - a) $({}^{-}2,2)$, $({}^{-}2,8)$; $({}^{-}7,4)$, $({}^{-}7,10)$
   - b) (1,1), (13,1), (10,5), (4,5)

REFERENCES:

Amsco - 490-500

Lesson 9

AIM: How do we find the midpoint of a line segment?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Derive the formulas for the abscissa and the ordinate of the midpoint of a line segment, given the coordinates of its endpoints.
2. Use the midpoint formulas to find the coordinates of the midpoint of any line segment, given the coordinates of its endpoints.
3. Used the midpoint formulas to find the coordinates of one end of a line segment given the coordinates of the midpoint and its other endpoint
4. Use the midpoint formulas to show specific relationships for given geometric figures.

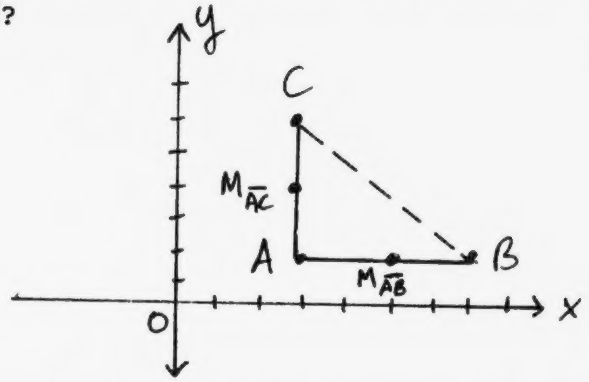
VOCABULARY: midpoint, abscissa, ordinate

MOTIVATION:

Given $\triangle ABC$ as $A(3,2)$, $B(7,2)$ and $C(3,6)$, what is the midpoint of $\overline{AB}$? of $\overline{AC}$?

DEVELOPMENT:

1. Plot points



By counting we see that (5,2) is the midpoint of $\overline{AB}$, and (3,4) is the midpoint of $\overline{AC}$. How can we find the midpoint of $\overline{BC}$?
Elicit various responses including first find the distance and then take $\frac{1}{2}$ of it or draw lines through the midpoints of $\overline{AB}$ and $\overline{BC}$ and parallel to the coordinate

axes thus forming two congruent right triangles, so that their point of intersection is the midpoint of $\overline{BC}$.
From this elicit that the point (5,4) has its abscissa midway between the two abscissas of the endpoints and the ordinate midway between the ordinates.
Therefore the formula is midpoint = $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.

2. Find the midpoint of $\overline{CD}$ when C(5,4) and D(-1,8), midpoint = $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}) = (\frac{5+{}^{-}1}{2}, \frac{4+8}{2}) = (\frac{4}{2}, \frac{12}{2}) = (2,6)$.

   Review: Addition of signed numbers and have the students find the midpoint of $\overline{RS}$ when R(-7,-2) and S(-9,5) and $\overline{PQ}$ when P(-5,-4) and Q(-2,3). Note: The midpoint of $\overline{PQ}$ will have a fractional ordinate.

3. Given $\overline{MN}$ with midpoint L(5,2) and M(9,5), find the coordinates of endpoint N.
   $Mx = \frac{x_1+x_2}{2}$ since we know the ordinates of the midpoint we get $5 = \frac{9+x}{2}$, 10 = 9+x, x = 1. Similarly for $My = \frac{y_1+y_2}{2}$, $2 = \frac{5+4}{2}$, 4 = 5+y and y = -1. Therefore point N is (1,-1).
   Find the coordinates of endpoint Q of $\overline{OR}$ if R(-5,6) and midpoint P(-7, -2). Elicit Mx=-7 therefore $-7 = \frac{-5+x}{2}$; -14 = -5 + x and x = -9 and My = -2. Therefore $-2 = \frac{6+y}{2}$; -4 = 6 + y and y = -10.

4. Find the center of a circle O, if diameter $\overline{AB}$ has endpoints A(9,2) and B(-1,0). Elicit the midpoint of $\overline{AB}$ is center O. Therefore point O = $(\frac{9+{}^{-}1}{2}, \frac{2+0}{2})$. O = $(\frac{8}{2}, \frac{2}{2})$ so Point O = (4,1).

5. Given vertices A(3,5), B(8,6), C(11,3) and D(6,2) show that ABCD is a parallelogram. Elicit that there are various ways to prove a quadrilateral is a parallelogram. Ask, "Can the midpoint formula be used in any of these?" Elicit if both diagonals $\overline{AC}$ and $\overline{BD}$ have the same midpoint, then they bisect each other. Therefore we find

$m\overline{AC} = (\frac{3+11}{2}, \frac{5+3}{2}) = (\frac{14}{2} + \frac{8}{2}) = (7,4)$ and $m\overline{BD} = (\frac{8+6}{2}, \frac{6+2}{2}) = (\frac{14}{2} + \frac{8}{2}) = (7,4)$. Therefore ABCD is a parallelogram since the diagonals bisect each other.

APPLICATIONS:

1. Find the coordinates of the midpoint of the segment joining each of the following points.
   a) A(4,6),B(2,2)   b) A(5,1) B(7,5)
   c) A(-2,5) B(4,-3)   d) A(-4,-1) B(-2,7).
2. Given that M is the midpoint of $\overline{LN}$, find the coordinates of N if
   a) M(2,6) and L(6,2)   b) M(-3,5) and L(1,-1)
   c) M(3,-4) and L(-5,6)   d) M(-6,-5).
3. Given the vertices of triangle QXR, find the coordinates of the midpoints of its sides.
   a) Q(1,1), X(3,7), R(9,3)
   b) Q(-2,3), X(0,5), R(6,-1)
4. Given the coordinates of the endpoints of diameter $\overline{ST}$ find the coordinates of point O the center of circle O.
   a) S(8,1), T(1,9)   b) S(-6,4), T(4,-8)
   c) S(3,-2), T(7,4)   d) S(-2,-1) T(6,-5)
5. Use the midpoint formula to show ABCD is a parallelogram if
   A(-2,2) B(-2,8) C(-7,4) D(-7,10)

SUMMARY:

1. How can we find the midpoint of an oblique line segment in coordinate geometry?
2. How do we find the center of a circle if we know the coordinates of the endpoints of a diameter?
3. Suppose the coordinates of the vertices of $\triangle RST$ are known. How can we find the length of the median drawn to side $\overline{ST}$?

HOMEWORK:

1. Find the coordinates of the midpoints of each of the following segments $\overline{JK}$.
   - a) J(8,2), K(8,6)
   - b) J(3,4), K(7,0)
   - c) J(-1,5), K(3,-3)
   - d) J(-2,-1), K(-7,3)
   - e) J(1,6), K(-3,2)

2. Given circle O and diameter $\overline{AB}$ find the coordinates of B if
   - a) O(10,10), A(2,2)
   - b) A(4,1), O(8,5)
   - c) O(-1,13), A(4,3)

3. Find the midpoint of the sides of $\triangle$LMO if L(3,5), M(-1,-3), O(7,1). What is the length of the median drawn to side $\overline{LO}$?

Lesson 10

AIM: How do we find the slope of a line?

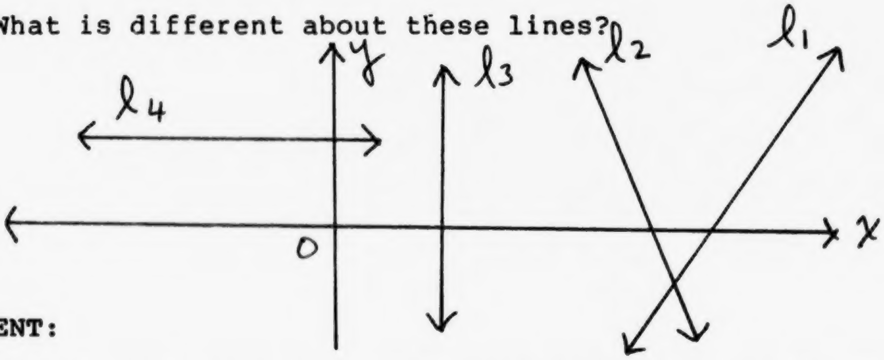
PERFORMANCE OBJECTIVES: The students will be able to...

1. Define "slope of a line" and write the formula for slope;
2. Find the slope of a line, given the coordinates of two points on the line;
3. Identify lines having different inclinations as having positive slope, negative slope, zero slope or no slope;
4. Find one coordinate of a point on a line given the other coordinate of the point and the slope of the line;
5. Use slope to prove that 3 points in a plane are collinear;
6. Through given points, draw lines having given slopes.

VOCABULARY: slope, incline, collinear.

MOTIVATION:

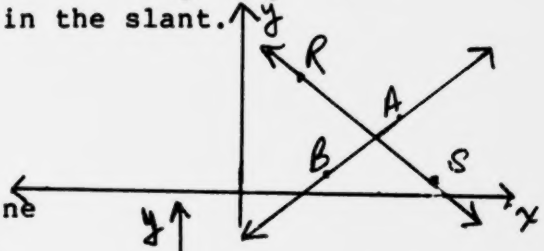
What is different about these lines?



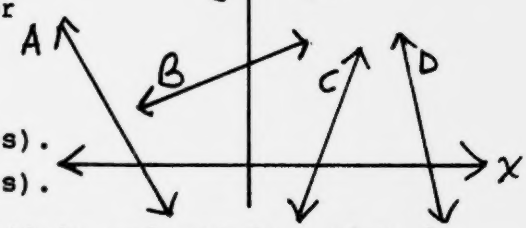
DEVELOPMENT:

1. Elicit that they have different "slants" or different inclinations or form different angles with the x-axis. We call this the "slope" of a line. Elicit the slope is a ratio of the change in Y to the change in X. The formula for finding the slope M, is $M = \frac{y_1-y_2}{x_1-x_2}$. $y_1 - y_2$ is sometimes called Δy (delta y) and x_1-x_2 is sometimes called Δx (delta x). Therefore the slope of a line is $M = \frac{\Delta y}{\Delta x}$.

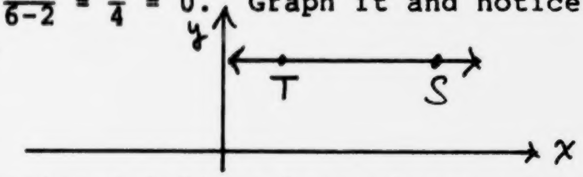
2. Find the slope of the line passing through points A(5,3) and B(3,1). Using the formula, we get $M = \frac{3-1}{5-3} = \frac{2}{2} = 1$. Therefore the slope is 1. Graph this line. Similarly find the slope of $\overleftrightarrow{RS}$ when R(2,5) and S(8,1). $M = \frac{5-1}{2-8}$. Note: Once you choose y_1, you must use the abscissa of the same point for x_1. Hence we get $M = \frac{4}{-6} = -\frac{2}{3}$. Plot this line. Note the difference in the slant. Ask, "Without finding it, will the slope of this line be positive or negative?"



3. Consider lines A,B,C,D. Determine whether they have positive or negative slopes.
Note:
(a and d have negative slopes).
(b and c have positive slopes).



4. Consider the line ST where S(6,3) and T(2,3). What is its slope? $M = \frac{3-3}{6-2} = \frac{0}{4} = 0$. Graph it and notice it is a horizontal line.



Ask, "Will all horizontal lines have slopes equal to 0? Why?" Yes, because the Y coordinates will be equal therefore the numerator will be 0, hence the slope is 0. Ask, "What will be the slope of a vertical line?" Elicit since the X coordinates are equal, the denominator will be equal to 0 and therefore it is undefined, that is, vertical lines have "no slope."

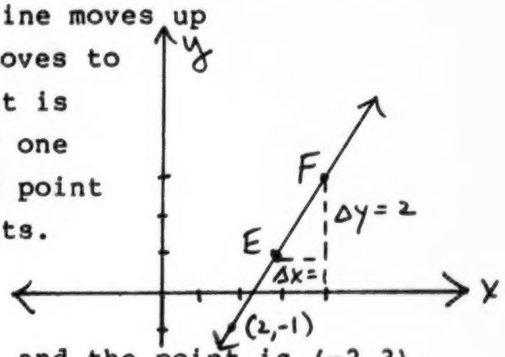
5. If line $\overleftrightarrow{JK}$ has a slope of $\frac{1}{2}$ and J(4,2) and K(X,10), can we find x? Use the formula $M = \frac{y_1-y_2}{x_1-x_2}$, $\frac{1}{2} = \frac{10-2}{x-4}$, cross multiplying we get 2(10-2) = 1(x-4); 16 = x-4 and x = 20. Let's try this again. Let $M = \frac{-2}{3}$, J(5,-3), K(X,5) we get $\frac{-2}{3} = \frac{5-(-3)}{X-5}$. Note $5-(-3) = 5 + {}^{+}3 = 8$.

Therefore we get $-\frac{2}{3} = \frac{8}{x-5}$, cross multiplying we get $3.8 = -2(x-5)$, $24 = 1-2x + 10$, $14 = -2x$ and $x = -7$.

6. Do points A,B,C form a straight line if A(-2,-12) B(2,3) and C(6,8)? How would we know without graphing?

Elicit if the slope of $\overleftrightarrow{AB}$ = slope of $\overleftrightarrow{BC}$ or $\overleftrightarrow{AC}$, then the points are collinear (lie on the same line). $m\overleftrightarrow{AB} = \frac{3-^{-}2}{2-^{-}2} = \frac{3+2}{2+2} = \frac{5}{4}$. $m\overleftrightarrow{AC} = \frac{8-^{-}2}{6-^{-}2} = \frac{10}{8} = \frac{5}{4}$ connecting the points we see in fact $\overleftrightarrow{ABC}$ is a straight line.

7. Can we use slope to draw a line. Consider $\overleftrightarrow{EF}$ with slope = 2 and E(3,1). What does slope = 2 mean? Elicit $\frac{\Delta y}{\Delta x} = \frac{2}{1}$. This means that the line moves up 2 units for every one unit it moves to the right. Hence a second point is (4,3) or down 2 units for every one it moves to the left so a third point is (2,-1). Now plot these points.



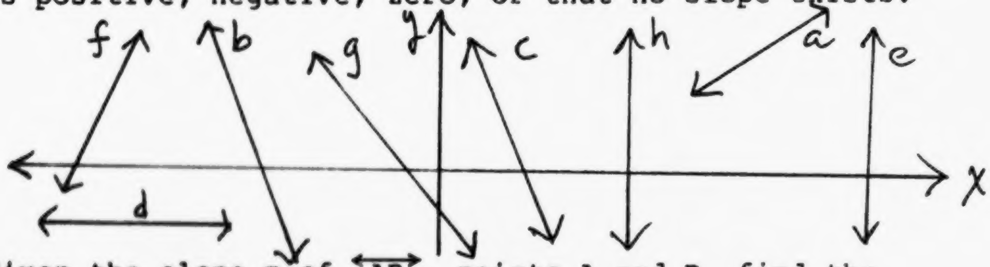
Let's try it if the slope is -1 and the point is (-2,3). Note a negative slope indicates opposite movement. That is up and left, down and right. Therefore other points are (-3,4) and (-1,2). How will we know which way the line goes? If as x increases, the line rises ($\Delta y > o$), then the line has a positive slope; if the line falls ($\Delta y < o$), then the line has a negative slope. Since the slope of this line is negative (-1), the line will fall.

APPLILCATIONS:

1. Find the slope of the lines containing the following points.

a)	(7,4),(3,2)	d)	(3,-2),(-2,4)
b)	(2,5),(5,6)	e)	(-4,-5),(4,-5)
c)	(5,1),(2,4)	f)	(2,3),(5,-3)

2. From the diagram determine whether the slope of each line is positive, negative, zero, or that no slope exists:



3. Given the slope m of $\overleftrightarrow{AB}$, points A and B, find the missing coordinate:

a) $m = 2$, A(4,6), B(2,y)   c) $m = \frac{-2}{3}$, A(4,y), B(-2,1)

b) $m = \frac{1}{2}$, A(x,3), B(1,5)   d) $m = 0$, A(4,-3), B(6,y)

4. Determine whether the three points are collinear:

a) (2,2), (4,7), (6,12)   c) (11,4), (-2,3), (-1,-1)

b) (7,9), (10,5), (13,1)   d) (-1,6), (-3,5), (-5,4)

5. Given a point and the slope m, graph the line, if:

a) $m = 2$, (2,1)   d) $m = \frac{2}{3}$, (4,-2)

b) $m = -2$, (3,2)   e) $m = \frac{-1}{2}$, (-3,2)

c) $m = \frac{1}{2}$, (2,-1)

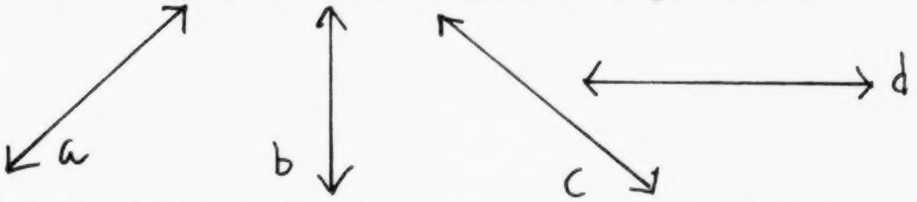
SUMMARY:

1. How do we find the slope of a line given the coordinates of two of its points?

2 How do we show points are collinear.

HOMEWORK:

1. Determine whether the slope of each line will be positive, negative, zero or that no slope exists.



2. Find the slope of the line joining the two points.

a) (4,3), (8,5)   c) (-2,3), (5,-2)

b) (7,2), (5,4)

3. Given the slope m of $\overleftrightarrow{AB}$ and points A and B, find the missing coordinate.

   a) $m = 1$, A(5,3), B(6,y) c) $m = -2$, A(5,y), B(-2,3)

   b) $m = \frac{1}{2}$, A(2,5), B(x,1)

4. Determine whether or not the three points are collinear.

   a) (2,1), (4,3), (6,5) b) (-3,4), (1,1), (1,-3)

5. Given the point A and the slope m, graph $\overleftrightarrow{AB}$, where A(2,3), and $m = 2$.

REFERENCES:

Amsco - 507-514

Lesson 11

AIM: How do we use slope to prove that lines are parallel or perpendicular?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Explain why two horizontal lines are parallel, two vertical lines are parallel and a vertical line and a horizantal line perpendicular to each other.
2. Prove that two non-vertical lines (oblique lines) are parallel if and only if their slopes are equal.
3. Define the term negative reciprocal.
4. Show that two non-vertical lines are perpendicular if and only if their slopes are negative reciprocals of each other.
5. Use these theorems to find slopes of lines parallel to given lines and slopes of lines perpendicular to given lines.
6. Use these theorems to show specific relationships for given geometric figures (eg. right triangles, parallelograms, rectangles, squares, rhombi, trapezoids)

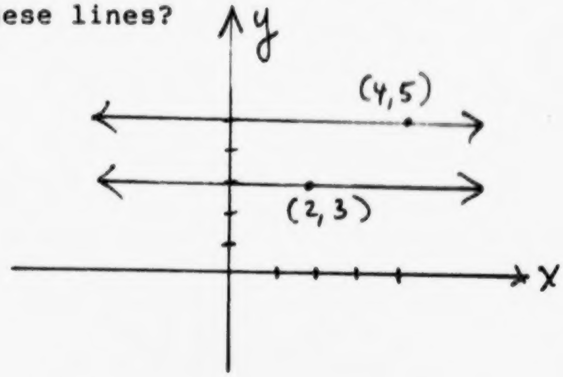
VOCABULARY: parallel, perpendicular, slope, negative reciprocal.

MOTIVATION:

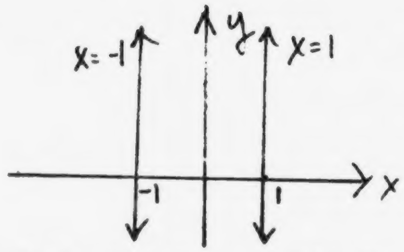
1. Draw the line with slope = 0 and passing through the point (2,3)
2. Draw line with slope = 0 and passing through the point (4,5) What is true about these lines?

DEVELOPMENT:

1. Elicit that the lines are parallel.

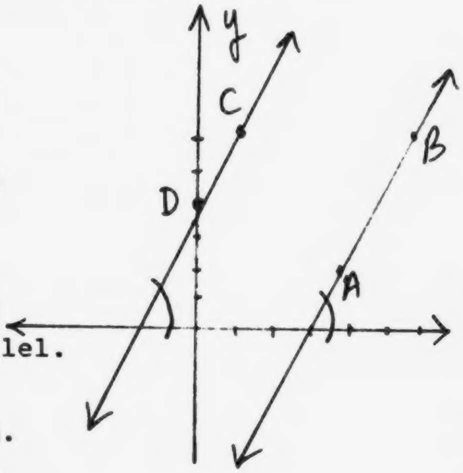


Ask, "Will this always be true?" Yes all horizontal lines are parallel. Are vertical lines such as X=1 and $X={}^{-}1$ parallel? Yes.



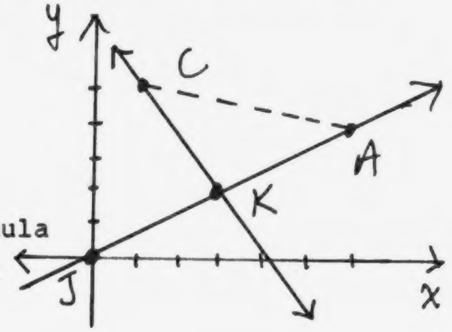
2. Consider the line AB determined by A(4,2) and B(6,6), and the line CD determined by C(1,6) and D(0,4). Are they parallel?

   Elicit that if they have the same slope, then they will make angles having the same measure with the X-AXIS and therefore be parallel by the theorem: If 2 lines are cut by a transversal and have congruent corresponding angles, then the lines are parallel. Will this always be true? Yes, therefore elicit the new theorem. If two lines (non-vertical) have equal slopes, then they are parallel. $m\overleftrightarrow{AB} = \frac{6-2}{6-4} = \frac{4}{2} = 2$ and $m\overleftrightarrow{CD} = \frac{6-4}{1-0} = \frac{2}{1} = 2$ therefore they are parallel. State the converse: If two non-vertical lines are parallel, then they will have the same slope. If $\overleftrightarrow{QR} \parallel \overleftrightarrow{ST}$ and the slope of $\overleftrightarrow{QR} = -\frac{1}{2}$ what is the slope of $\overleftrightarrow{ST}$ ($-\frac{1}{2}$, since the lines are $\parallel$, they must have = slopes).



3. Consider $\overleftrightarrow{JA}$ and $\overleftrightarrow{CK}$ where J(0,0), A(6,4), C(1,5) and K(3,2).

   Ask, "What appears to be true about the lines?" They appear to be $\perp$ (perpendicular) Could we prove this? Yes, draw $\overline{AC}$ and show by the distance formula and the Pythagorean Theorem that $\triangle$ACK is a right $\triangle$.



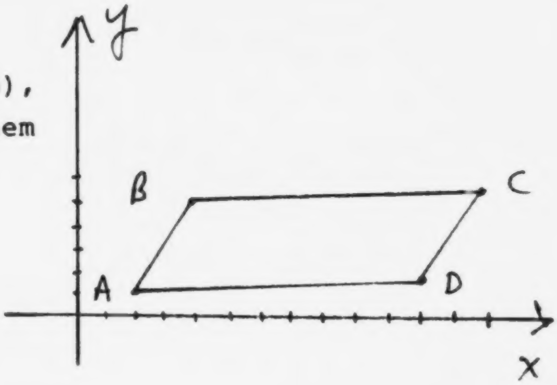
4. Assign this for homework or extra credit. There is another way. Find the slopes of the two lines $m\overleftrightarrow{JA} = \frac{4-0}{6-0} = \frac{4}{6} = \frac{2}{3}$, $m\overleftrightarrow{CK} = \frac{3}{-2} = \frac{-3}{2}$

Is there a relationship? Yes, they are reciprocals with opposite signs. These are called negative reciprocals. Will $\perp$ lines always have slopes that are negative reciprocals? (yes) Will the converse be true? (yes) Elicit the following theorem and its converse:

If two lines (non-vertical) have slopes that are negative reciprocals of each other, then they are perpendicular. (Converse: If two non-vertical lines are perpendicular, then their slopes are negative reciprocals.)

5. What is the slope of a line perpendicular to another line whose slope is $\frac{1}{2}$? $(-\frac{2}{1} = {}^{-}2.)$ If the slope is $\frac{-5}{4}$, what will be the slope of a line $\perp$ to it? $(\frac{4}{5})$

6. What can we prove about quadrilaterals and triangles using slope?
   Elicit - we can show:
   a) a $\triangle$ is a right $\triangle$ by showing slopes of two sides are negative reciprocals.
   b) a quadrilaterial is a parallelogram by showing that the opposite sides have the same slope and, therefore, are parallel.
   c) a parallelogram is a rectangle by proving it contains a right angle by showing that the slopes of a pair of adjacent sides are negative reciprocals.
   d) a parallelogram is a rhombus by showing that its diagonals are perpendicular.



Consider points A(2,1), B(4,5), C(13,6) and D(11,2). Plot them and show that they form a parallelogram.

$m(\overline{AB}) = \frac{5-1}{4-2} = \frac{4}{2} = 2,\ m(\overline{CD}) = \frac{6-2}{13-11} = \frac{4}{2} = 2$

Therefore $\overline{AB} \parallel \overline{CD}$

$m(\overline{BC}) = \frac{6-5}{13-4} = \frac{1}{9}\quad m(\overline{AD}) = \frac{2-1}{11-2} = \frac{1}{9}$

Therefore $\overline{BC} \parallel \overline{AD}$ Hence ABCD is a ▱ since opposite sides are $\parallel$.

APPLICATIONS:

1. Find the slope of a line parallel to the $\overleftrightarrow{AB}$ if the slope of $\overleftrightarrow{AB}$ is
   a) 2 b) $-\frac{1}{2}$ c) -4
2. What is the slope of a line perpendicular to $\overleftrightarrow{RS}$ if the slope of $\overleftrightarrow{RS}$ is
   a) 2 b) $-\frac{1}{2}$ c) $\frac{2}{3}$ d) $-\frac{3}{4}$ e) -5
3. Are the following lines, $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ $\parallel$, $\perp$, or neither given two points on each line.
   a) A(3,5), B(7,3); D(2,2), E(6,0)
   b) A(2,3), B(-2,4); D(1,4), E(0,0)
   c) A(-2,-3), B(3,2); D(3,1), E(5,-1)
4. Show that $\triangle$ ABC is a right $\triangle$ if A(-1,4), B(3,4), C(3,2)
5. What type of special quadrilateral is ERIC if E(2,1), R(7,2), I(9,6), C(4,5)
6. Lines $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ of A(5,2), B(2,1), C(-1,Y), D(2,4) Find Y.
7. Show that HERB is a Rhombus if H(5,1), E(8,3), R(2,3), B(5,5)

SUMMARY:

1. How can we show that lines are parallel in the coordinate plane?
2. How can we show that two lines are perpendicular?

HOMEWORK:

1. Find the slope of a line $\perp$ to line e if the slope of line e is.
   a) 3 b) $\frac{1}{5}$ c) -4 d) $-\frac{3}{2}$
2. Find the slope of a line $\parallel$ to the line joining the given points.
   a) (4,2), (1,5) b) (3,3), (1,2) c) (-3,1), (2,-3)
3. Three vertices of a parallelogram are (-4,2), (-2,-4), (6,3) in order. What are the coordinates of the fourth vertex?
4. What type of special triangle is ABC if A(2,2), B(1,3), C(7,7)?
5. If the vertices of parallelogram ABCD are A(-2,2), B(2,4), C(7,0), D(4,Y) find Y.

REFERENCES:

Amsco 515-525

Lesson 12

AIM: How do we graph a linear equation?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Define the graph of an equation as the graph of the solution set of that equation.
2. State as a postulate that the graph of every first degree equation in two variables of the form $ax+by+c = 0$ where a and b are not both zero, is a straight line and its converse.
3. Explain the meaning of x-intercept and y-intercept of a graph.
4. Explain why first degree equations in two variables are also called linear equations.
5. Draw graphs of linear equations, including those whose graphs are parallel to the x-axis or to the y -axis.
6. Determine intercepts of lines drawn in a plane.
7. Determine whether a given point lies on a given line.

VOCABULARY: linear equation, x-intercept, y-intercept, first-degree equation, solution set.

MOTIVATION:

Warm-up: Find 3 ordered pairs that are solutions for the equation $y = 2x+1$.

DEVELOPMENT:

1. Elicit 3 ordered pairs such as (1,3), (2,5), (0,1) Ask, "What do these appear to be?" Elicit they could represent the coordinates of points in the Cartesian plane. Let's plot and connect them. What do you see? (The points determine a straight line). Let's try this with another equation,

y = 3x -4. Recall we can use a table to find values that will satisfy this equation.

Have the students plot these at their seats. Send one student to the board to plot and connect them. Again elicit that the points form a straight line. Ask, "Will this always be true?" (yes)

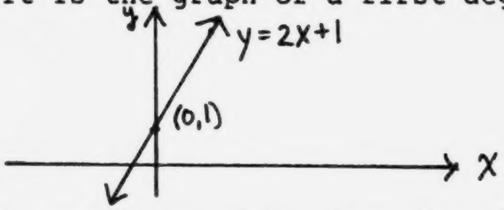
X	3x -4	y	point
1	3(1)-4	-1	(1,-1)
0	3(0)-4	-4	(0,-4)
-13	(-1)-4	-7	(-1,-7)

Definition: The graph of an equation is the graph of the set of all ordered pairs of numbers that satisfy the equation.

We postulate that the graph of every first degree equation is a straight line.

Will the converse be true? (yes) Elicit the converse.

Postulate: If the graph of an an equation is a straight line then it is the graph of a first degree equation.

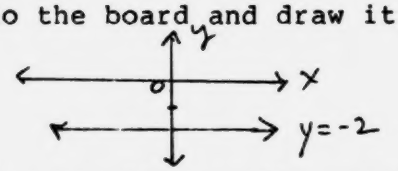


2. Consider the graph y = 2x+1, (see above). Where does it cross the y-axis? (at y = 1, point (0,1) This is called the y-intercept. Elicit, Definition : The y-intercept of a line is the point of intersection of the line and the y-axis. What is the y-intercept of y = 3x-4? (The point (0,4).)

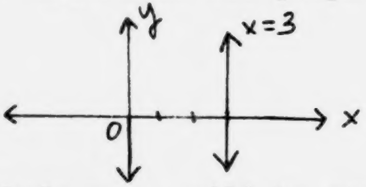
   Ask: What would an x-intercept be? Elicit, Definition : The x-intercept is the point where the line crosses the x-axis. Consider the equation y = x+3. What would be the y-intercept? Elicit (0,3). We get the y-intercept by letting x = 0 and finding y. What would be the x-intercept? ((0,$^{-}$3), since 0 = x+3 so that x = -3.)

3. Consider the equation y = 3. Ask what points will satisfy this equation? Elicit, (1,3), (2,3), (8,3), (4,3) etc. Graph these and see what you get.

Ask: What would the line $y = {}^{-}2$ look like? Have a student go to the board and draw it.



Elicit, a line parallel to the x-axis has an equation of the form $y = a$. What will the graph of $x = 3$ look like? Elicit,



Similarly, conclude $x = a$ will be parallel to the y-axis.

4. Consider the line whose equation is $x+y = 7$. Is the point (1,6) on the line? How do we know? By substitution, $1+6 = 7$, therefore the point is on the line because it satisfies the equation. Is the point $({}^{-}2,3)$ on the line whose equation is $2x + 3y = 5$? Yes, since $2(-2) + 3(3) = -4+9 = 5$.

APPLICATIONS:

1. Find points and draw the graph of each of the following equations.
   a) $y = 3x-2$
   b) $2x + 2y = 6$
   c) $x = 2y-1$

2. Draw the graphs of each of the following.
   a) $y = 2$ b) $x = 5$ c) $x = {}^{-}2$ d) $y = {}^{-}3$

3. Find the y-intercept of each of the following.
   a) $y = 2x + 2$
   b) $y = \frac{1}{2}x - 3$
   c) $2y = 4x - 6$
   d) $2x + 3y = 9$
   e) $x - 2y = 8$

4. Find the x-intercept of each of the following.
   a) $x = y + 3$
   b) $y = 2x -4$
   c) $x + y = 7$
   d) $2x - 2y = 2$
   e) $2x = 3y$

5. Determine if the given point lies on the given line.
   - a) $(4,2)$, $y = x-2$ c) $(6,1)$, $y - x = 5$
   - b) $({}^-3,{}^-2)$, $2y = 3x+5$ d) $2x - 4y = 4$, $(4,1)$

6. Find the value of p so that the given point lies on the given line.
   - a) $x + y = p$, $(5,-7)$
   - b) $2x - y = p$, $(2,3)$
   - c) $y - 2x = p$, $(4,-1)$

SUMMARY:

1. How do we graph a line given its equation?
2. How do we find the x - intercept and y - intercept of a line from its equation.
3. What is the equation of a line that passes through the point $P(-2,3)$ and is (a) parallel to the y-axis? (b) perpendicular to the y-axis?

HOMEWORK:

1. Graph each of the following equations.
   - a) $y = x + 5$ b) $2y = x - 3$ c) $x + 2y = 5$
2. Find the x-intercept and y-intercept of each of the following.
   - a) $2x + 3y = 6$ b) $y = x\,{}^-7$ c) $4x - 2y = 3$
3. a) $(4,4)$, $y = 2x - 4$ c) $({}^-2, {}^-1)$, $2x - y = 2$
   - b) $(3,1)$, $x + 2y = 5$ d) $({}^-3,1)$, $x + 2y = {}^-1$

REFERENCES:

Amsco 525-528

Lesson 13

AIM: How do we write the equation of a straight line?
(point-slope form)

PERFORMANCE OBJECTIVES: The students will be able to...

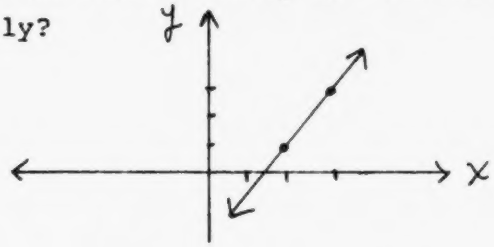
1. Derive the equation of a line, given the slope of the line and the coordinates of one point on that line. (point-slope form)
2. Find the equation of a line given the coordinates of two points on the line.
3. Write equations and draw graphs of lines, given their slopes and the coordinates of a point on the line
4. Write the equation of a line parallel to a given line and passing through a given point, using the point-slope form.
5. Write the equation of a line perpendicular to a given line and passing through a given point, using the point-slope form.

VOCABULARY:

equation, parallel, point-slope form

MOTIVATION:

Graph the line whose slope = 2 and which passes through the point (2,1). If we can draw a line, how can it be represented algebraically?



DEVELOPMENT:

1. Elicit that the line is the graph of an equation. Consider the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$. Since we know the

slope = 2 and one point is (2,1), we need a second point, call it (x,y). Hence, we have $2 = \frac{y-1}{x-2}$. This is a valid equation. If we wish we could clear the fraction and get $2(x-2) = y-1$ which simplifies to $2x - 4 = y-1$, $y = 2x - 3$. Example: Find the equation of the line with $m = \frac{1}{2}$ and passing through the point (-1,3). Using the slope formula, $\frac{1}{2} = \frac{y-3}{x+1}$ which becomes $\frac{1}{2}(x+1) = y - 3$. Simplifying, $x + 1 = 2y - 6$ or $x+7 = 2y$ or $y = \frac{x}{2} + \frac{7}{2}$.

2. Suppose we have a line passing through the points (2,5) and (3,1), how would we find its equation? Elicit, first find the slope and then use one of the points. That is, $m = \frac{5-1}{2-3} = \frac{4}{-1} = -4$. Now substitute (x,y) for one point and we get $-4 = \frac{y-5}{x-2}$. Simplifying, $-4(x-2) = y - 5$, $-4x + 8 = y-5$ or $-4x + 13 = y$.

3. Consider the line whose equation is $x+y = 7$, what would be the equation of the line parallel to this line and passing through the point (2,1)? First find the slope of the given line. Pick two points (3,4), (5,2) and find m. $m = \frac{2-4}{5-3} = \frac{-2}{2} = -1$. Recall that lines are parallel if they have the same slope. Therefore we proceed as before and write $-1 = \frac{y-1}{x-2}$ which equals $-(x-2) = y-1$ or $-x+2 = y-1$ or $x+y = 3$.

4. Suppose we wanted to find the equation of a line perpendicular to the line $x+y = 3$ and passing through the point (4,4). How would we proceed? Elicit that if lines are perpendicular; then their slopes are negative reciprocals. What was the slope of the line $x+y = 3$? (-1) What is its negative reciprocal? (+1) Therefore, the equation is $1 = \frac{y-4}{x-4}$ or $x-4 = y-4$ or $x = y$.

5. Draw the graph of a line parallel to $y = 2x + 2$ and passing through the point (2,3). First find the slope of $y = 2x + 2$. (2) This gives us the equation of the new line as $2 = \frac{y-3}{x-2}$ or $2x-4 = y-3$ or $2x-1 = y$. What additional information do we need in order to graph this line? (Find the coordinates of a second point that the line passes through.) Letting $x=1$, $y = 2(1)-1 = 1$.

Points (2,3) and (1,1) are on the desired line. Next, have students graph the line.

Note: you also might want to have a student graph $y = 2x+2$.

MEDIAL SUMMARY:

Elicit that, in general, if a line has slope and passes through the point (x_1, y_1) then its equation can be represented as $m = \frac{y-y_1}{x-x_1}$ or $y-y_1 = m(x-x_1)$.

APPLICATIONS:

1. Write the equation of a line with the indicated slope and containing the indicated point.
   - a) $m = 1$, $(2, 3)$
   - b) $m = \frac{1}{2}$, $(4, 2)$
   - c) $m = -2$, $(-1,-2)$
2. Write the equation of line $\overleftrightarrow{AB}$ with point A and B as follows.
   - a) $A(4,1)$, $B(2,4)$,
   - b) $A(-2,5)$, $B(5,1)$
   - c) $A(-7,8)$, $B(4,-10)$
3. What is the equation of a line parallel to the line $2x+4 = y$ and passing thru the point $(1,4)$.
4. Given the equation $y+x = 5$, write the equation of a line.
   - a) parallel to it and passing through point $(2,-3)$
   - b) perpendicular to it on passing through point $(7,2)$
5. Determine whether the following lines are parallel, perpendicular or neither.
   - a) $y = 2x-5$ and $2y = 4x-5$
   - b) $x+y = 7$ and $x-y = 3$
   - c) $3x+2y = 5$ and $2x-3y = 8$
   - d) $x+y = 5$ and $2x+y = 5$

SUMMARY:

1. How do we write the equation of a line given a point and its slope?
2. How do we find the equation of a line given two points on the line?

HOMEWORK:

1. Write the equation of a line with the indicated slope and containing the indicated point.
   a) $m = 2$, $(4,5)$　　b) $m = -\frac{1}{3}$, $(2,1)$
   c) $m = \frac{3}{5}$, $(-2,3)$
2. Write the equation of line $\overleftrightarrow{AB}$ with points A and B as follows.
   a) $A(3,1)$, $B(2,4)$
   b) $A(-2,3)$, $B(5,-1)$
   c) $A(-2,-4)$, $B(1,-2)$

3 What is the equation of a line perpendicular to the line $2x-2y = 4$ and passing through the point $(1,-2)$.

4. Write the equation of the line parallel to the line $y = 3x-2$ and passing through the point $(2,1)$.

Lesson 14

AIM: How do we write the equation of a straight line? (slope-intercept form)

PERFORMANCE OBJECTIVES: The students will be able to...

1. Derive the equation of a line, given the slope and y-intercept of the line from the point-slope form.
2. Find the slope of a line, given the coordinates of two points on the line.
3. Determine the slope and y-intercept of a line from the equation of the line.
4. Determine whether two lines whose equations are given are parallel or whether they are perpendicular to each other.
5. Write the equations of lines having given y-intercepts and either parallel or perpendicular to a given line.

VOCABULARY: slope-intercept, y-intercept, coefficient

MOTIVATION:

Find the slopes of the following lines:
a) $y= 2x+3$ b) $y = 3x-2$ c) $y = -3x+1$

DEVELOPMENT:

1. The students should find the slopes by finding two points and using the slope formula. From this note the slope of the first equation is 2, of the second is 3 and of the third is -3.
   Elicit that these are the coefficient of x in each equation. Without using the slope formulas what is the slope of the line whose equation is $y = 4x-2$? (4) $y = \frac{1}{2}x-1$? ($\frac{1}{2}$)
2. Is this always true? Let's see. Find the slope of $y + 2x = 1$. Get points, such as (1,-1) and (-2,5). Using these points, substitute into the slope formula:

$m = \frac{5-^{-}1}{-2-1} = \frac{6}{-3} = -2$. This is not the coefficient of x but it has the same absolute value. If we rewrite the equation as y = ..., we get y = -2x+1 and there is our slope. Elicit that when an equation is expressed in the form y = mx+b, the coefficient of x is the slope (m).

3. What are the y-intercepts of these equations?
Elicit these are 3 for #1, -2 for #2 and 1 for #3.
That is, the b term in y = mx+b is the y-coordinate of the y-intercept of the line defined by this equation. This is called the slope-intercept form of an equation of a line.

4. Find the slope and y-intercept of each of the following.
   a) y = 3x-2, (m = 3, b = -2)
   b) 2x+2y = 6. What must we do? Express it in y = mx+b form. Algebraically, 2y = -2x+6 and dividing by 2 we get y = -x+3.

   Elicit that the coefficient of x is -1. Therefore, the slope is -1 and the y-intercept is 3.
   c) 3x+2y = -8, again we need the slope-intercept form of the equation.

   Elicit that we get 2y = -3x-8 or $y = \frac{-3}{2}x-4$. Therefore, the slope is $\frac{-3}{2}$ and the y-intercept is -4.

5. Can we show that this is true for all lines? Let's begin with a line of slope m and y-intercept (0,b). Using the original slope formula we get $m = \frac{y-b}{x-0}$ or mx = y-b or mx+b = y which is y = mx+b.

6. Are the lines y = 2x+4 and 4x-2y = 2 parallel or perpendicular to each other? Elicit that we need their slopes. The slope of the first line is 2. For the second line we get 4x=2y+2, 4x-2=24 and y =2x-1.
This also has a slope of 2 so the lines are parallel.

7. Write the equation of the line with m = 2 and passing through (1,4). Can we use y = mx+b? Yes, if we find b.
There are 4 variables in this equation, how many do we know? Then since =2, x=1, and y=4. Substituting in y = mx+b, we

get: $y = mx+b$

$4 = 2(1) + b$

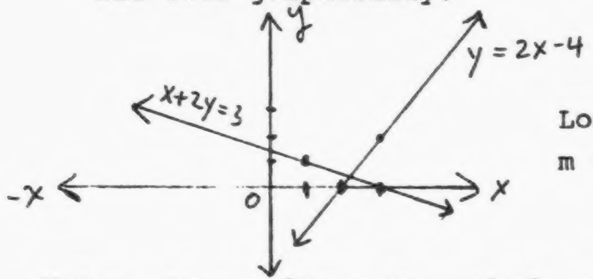
$4 = 2 + b$

$2 = b$ Therefore the equation is $y = 2x+2$

8. Write the equation of a line perpendicular to $x+2y = 3$ and passing through the point (3,2). To do this we must first find the slope of the given line. That is, express it in slope-intercept form: $x+2y = 3$ gives $2y = -x+3$ or $y = -\frac{1}{2}x+\frac{3}{2}$. Therefore, $m = -\frac{1}{2}$. What is the slope of the new line? The negative reciprocal of $-\frac{1}{2}$ which is 2.

Therefore we have:
$y = mx+b$
$2 = 2(3)+b$
$2 = 6+b$
$-4 = b$

and the equation of the perpendicular is $y = 2x-4$. We can see this graphically.



Locate the point (3,2) and use $m = 2$.

Note: As an alternate method, students may find it easier to use the point-slope form of the equation of a line.

APPLICATIONS:

1. Find the slope and y-intercept of each of the following:
   a) $y = 2x-3$ b) $3x-y = 5$ c) $x+2y-3 = 0$

2. Write the equation of the line with the indicated slope and containing the given point.
   a) $m = 1$, (0,5)
   b) $m = \frac{1}{2}$, (1,3)
   c) $m = -2$, (5,-2)

3. Write the equation of a line parallel to the given line and containing the given point.
   a) $y = 2x-1$, (2,2) c) $x+y = 5$, (-1,5)
   b) $y = \frac{1}{2}x+3$, (4,-2) d) $2x-y = 8$, (-4,-3)

4. Write the equation of the line perpendicular to the given line and containing the given point.
   a) $y = 3x-2$, $(1,3)$ c) $2x-y = 4$, $(2,-3)$
   b) $y = \frac{-2}{3}x+4$, $(-3,-5)$ d) $3x+4y-2 = 0$, $(-2,4)$

5. Determine whether the lines are parallel, perpendicular or neither.
   a) $y = 2x+3$ and $y = 2x-3$ c) $2x+3y = 5$ and $3x-2y = 4$
   b) $y = 4x+3$ and $y = \frac{-1}{4}x-1$ d) $5x+3y-2 = 0$ and $y=5x-1$

SUMMARY:

1. What is the slope-intercept form of the linear equation?

2. What is the relationship between the slopes of lines that are parallel? perpendicular? intersecting?

3. For what value of p are the lines parallel? perpendicular?
   $2y+3x= 8$
   $y+px= 5$

HOMEWORK:

1. Find the slope and y-intercept of each of the following.
   a) $y = x+5$ b) $2y+x = 9$ c) $3x+5y-2 = 0$

2. Write the equation of the line with the indicated slope and containing the given point.
   a) $m = 2$, $(0,2)$
   b) $m = -3$, $(1,-3)$
   c) $m = \frac{2}{3}$, $(-3,4)$

3. Write the equation of the line parallel to the given line and passing through the given point.
   a) $y = 3x-2$, $(4,1)$ b) $x+y+4 = 0$, $(2,-3)$

4. Write the equation of the line perpendicular to the given line and containing the point.
   a) $y = 4x+1$, $(2,3)$ b) $2y-x = 3$, $(-1,5)$

Lesson 15

AIM: How do we show specific relationships for given geometric figures using coordinate geometry?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Prove that a polygon, given the coordinates of its vertices, is a specific type of triangle (equilateral, isosceles, scalene, right) or a specific type of quadrilateral (parallelogram, rectangle, rhombus, square, trapezoid, isosceles trapezoid) by using the distance formula, midpoint formulas, slope, properties of parallel and perpendicular lines, and equations of lines.
2. Prove that the altitude to the base of an isosceles triangle, given the coordinates of its vertices, is also the median to the base.
3. Prove that a line segment joining the midpoint of two sides of a triangle, given the coordinates of its vertices, is parallel to the third side and equal to half its length.
4. Prove that the diagonals of a quadrilateral, given the coordinates of its vertices, bisect each other and/or are congruent and/or perpendicular to each other.

VOCABULARY: vertices, diagonals

MOTIVATION:

1. Plot points A(1,1), B(5,1), C(1,4) connect the points. What kind of figure is drawn? (Right $\triangle$) How can we prove this result?

DEVELOPMENT:

1. We could prove that $\triangle$ ABC is a right triangle in two ways. Elicit that these are to show it contains a right angle or that it satisfies the Pythagorean Theorem. How could we do this using coordinate geometry?

Elicit - a) using slope we could show two sides are perpendicular or

b) using the distance formula we could show that the length of the sides satisfy $a^2+b^2= c^2$.

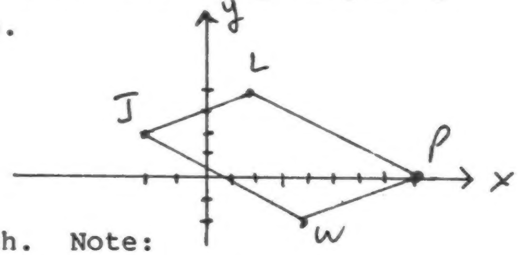
Consider QRS with vertices Q(2,3), R(5,7) and S(2,8). What kind of triangle is it? First plot the points. What does it look like? Elicit it could be isosceles. How could we find out? Use the distance formula.

QS = 5 (by counting) Find $d\overline{QR}$.

$d\overline{QR} = (7-3)^2+(5-2)^2 = 16+9 = 25 = 5$

Therefore the triangle is isosceles.

2. Consider quadrilateral WPLJ with W(4,-2), P(8,0), L(2,4) and J(-2,2). What kind of quadrilateral is it? Again we plot the points and draw it. It appears that it could be a ▱. How could we show this? Elicit by comparing either slopes or distances.



Have half the class do each. Note:

by slope $m\overline{WP} = \frac{2}{4} = \frac{1}{2}$ $m\overline{JL} = \frac{2}{4} = \frac{1}{2}$, so $\overline{WP} \parallel \overline{JL}$

$m\overline{PL} = \frac{-4}{6} = \frac{-2}{3}$ $m\overline{JW} = \frac{4}{-6} = \frac{-2}{3}$, so $\overline{PL} \parallel \overline{JW}$. Therefore, it is a ▱ since opposite sides are parallel.

By the distance formula we get $d\overline{WP} = \sqrt{(8-4)^2+(0-\ 2)^2} = \sqrt{16+4} = \sqrt{20}$; $d\overline{JL} = \sqrt{(4-2)^2+(2-{}^-2)^2} = \sqrt{2^2+4^2} = \sqrt{20}$, $d\overline{PL} = \sqrt{(8-2)^2+(0-4)^2} = \sqrt{36+16} = \sqrt{52}$ and $d\overline{JW} = \sqrt{(4-{}^-2)+({}^-2{}^-2)^2} = \sqrt{36 + 16} = \sqrt{52}$. Therefore WPLJ is a ▱ since the opposite sides are equal.

3. Could we use coordinate geometry to show that the diagonals $\overline{JP}$ and $\overline{LW}$ bisect each other? Elicit that if they have the same midpoint, they bisect each other. For $\overline{JP}$, $M_x = \frac{8+-2}{2} = \frac{6}{2} = 3$ $M_y = \frac{0+2}{2} = 1$ Therefore the midpoint of $\overline{JP}$ is (3,1). The midpoint of $\overline{WL}$ is $M_x = \frac{6}{2} = 3$ $M_y = \frac{-2+4}{2} = \frac{2}{2} = 1$ is (3,1).

4. Given the coordinates of the vertices of quadrilateral BECK as B(0,3), E(4,6), C(7,2), K(3,-1)

   a) Show BECK is a rhombus.
   b) Show it is a square. (How can we show that it is a square?)

Elicit that there are many ways to do this. One is to find the length of the 4 sides and show they are equal. Another is to show that the opposite sides are parallel and that the length of 2 adjacent sides are equal. But the best way would be to show that the diagonals bisect and are perpendicular to each other. The midpoint of diagonal $\overline{BC} = (\frac{7}{2}, \frac{5}{2})$ = the midpoint of diagonal $\overline{EK}$. Therefore the diagonals bisect each other and BECK is a parallelogram. To show that $\overline{BC} \perp \overline{EK}$ find and compare their slopes.

$m\overline{EK} = \frac{6-{}^{-}1}{4-\ 3} = \frac{7}{1} = 7$, $m\overline{BC} = \frac{3-2}{0-7} = -\frac{1}{7}$

Hence, $\overline{BC} \perp \overline{EK}$. BECK is a rhombus since it is a parallelogram whose diagonals are perpendicular to each other.

To show that rhombus BECK is a square all we need do is show that a pair of adjacent sides are perpendicular.

$m\overline{BE} = \frac{6-3}{4-0} = \frac{3}{4}$, $m\overline{EC} = \frac{6-2}{4-7} = -\frac{4}{3}$

Hence, $\overline{BC} \perp \overline{EK}$ and BECK is a square.

5. Is a median of an isosceles or equilateral triangle also an altitude? How can we show this using coordinate geometry. Let's consider ΔABC with A(0,⁻1), B(4,7) and C(6,1), how do we know ΔABC is isosceles?

$d\overline{AC} = \sqrt{(6-0)^2+(1-\ 1)^2} = \sqrt{36 + 4} = \sqrt{40}$

$d\overline{BC} = \sqrt{(6-2)^2+(7-\ 1)^2} = \sqrt{4-36} = \sqrt{40}$

To find the median we find the midpoint of $\overline{AB}$, $mx = \frac{0+4}{2} = 2$, $my = \frac{-1+7}{2} = \frac{6}{2} = 3$. D=(2,3) is the midpoint of AB. Is this also an altitude? Find the slope of $\overline{AB}$, $m = \frac{7-\ 1}{4-\ 0} = \frac{8}{4} = 2$, and the slope of $\overline{CD}$, $m = \frac{3-1}{2-6} = \frac{2}{-4} = \frac{-1}{2}$. Therefore the lines are $\perp$.

APPLICATIONS:

1. Show that $\triangle SAT$ is a right triangle given S(5,4), A(2,3), T(1,6).
2. Given quadrilateral ABCD with A(0,1), B(5,3), C(7,-2) and D(2,-4)
   - a) Show that ABCD is a Rhombus.
   - b) Show that ABCD is a square.
3. Using only one pair of opposite sides show that CHER is a parallelogram given C(-1,2), H(3,4), E(2,-1) and R(-2,-3)
4. Given quadrilateral BECK with B(-1,0), E(1,2), C(4,2) and K(-1,-3), show that BECK is an isosceles trapezoid.
5. Given $\triangle UPS$ with vertices U(0,-2), P(4,6) and S(6,0) show
   - a) $\triangle UPS$ is an isosceles triangle.
   - b) Show that the altitude to the base is also the median.

SUMMARY:

1. Describe 3 different methods of proving a quadrilateral is a parallelogram using coordinate geometry.
2. Describe 2 different methods of proving a triangle is a right triangle using coordinate geometry.
3. Using only the distance formula, how would you prove that a quadrilateral is a rectangle?

HOMEWORK:

1. Given $\triangle PLJ$ with vertices P(0,4), L(3,8) and J(7,5), show that $\triangle PLJ$ is an isoceles right triangle.
2. Quadrilateral WXYZ has W(1,0), X(3,3), Y(8,0) and Z(7,-3) as its vertices. Show that WXYZ is a parallelogram.
3. Show that the midpoints of the sides of quadrilateral STAN form a parallogram if the vertices of STAN are S(1,5), T(9,9), A(13,1) and N(3,-1).

4 Given points C(-4,2), A(0,3), R(4,1) and M(-4,-1), show that CARM is a trapezoid but not an issosceles trapezoid.

Lesson 16

AIM: How do we prove general theorems, using coordinate geometry?

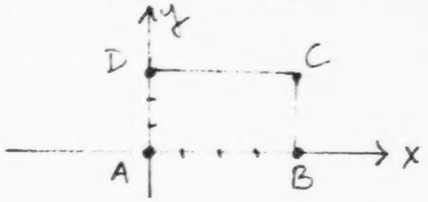
PERFORMANCE OBJECTIVES: The student will be able to...

1. Use the distance formula, midpoint formulas, slope, properties of parallel and perpendicular lines, and equations of lines to prove theorems related to polygons whose vertices have coordinates stated in general terms by using variables.
2. Use coordinate geometry to prove theorems from Euclidean geometry related to triangles and quadrilaterals (that is, "If a line bisects two sides of a triangle, then the line is parallel to the third side," or "The diagonals of a rectangle are congruent.")

VOCABULARY: vertices, coordinates

MOTIVATION:

1. Warm-up: Show that the diagonals of rectangle ABCD with vertices A(0,0), B(4,0), C(4,3), D(0,3) are congruent. To show this we show that d $\overline{AC}$ = d $\overline{BD}$.

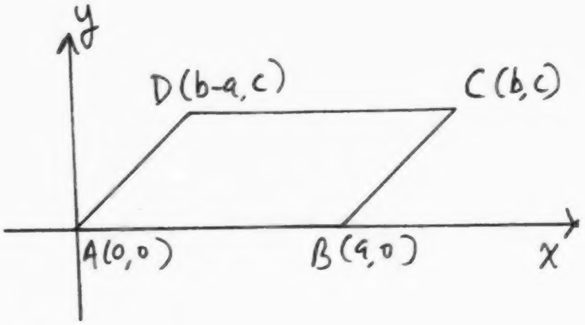


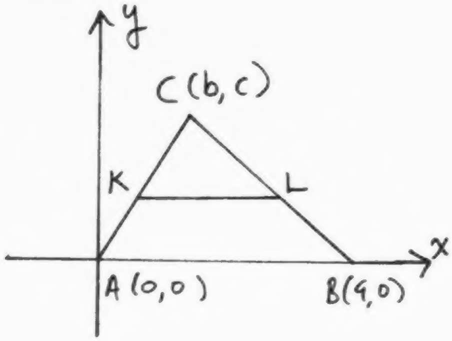
DEVELOPMENT:

1. d $\overline{AC}$ $=\sqrt{(4-0)^2 + (3-0)^2} = 5$, d $\overline{BD}$ $=\sqrt{(4-0)^2 + (0-3)^2} = 5$. Could we prove this relationship for all rectangles? Yes, by using a general rectangle. Let's consider rectangle ABCD with coordinates A(0,0), B(a,0), C(a,b), D(0,b).

To show the diagonals are congruent, find the lengths of $\overline{AC}$ and of $\overline{BD}$. $d\overline{AC} = \sqrt{(a-0)^2+(0-b)^2} = \sqrt{a^2+b^2}$. And $d\overline{BD} = \sqrt{(0-a)^2+(0-b)^2} = \sqrt{a^2+b^2}$. Therefore the diagonals of a rectangle are congruent.

2. Prove that the diagonals of a parallelogram bisect each other. Consider parallelogram ABCD with A(0,0), B(a,0), C(b,c), D(b-a,c). Why are the coordinates of D(b-a,c)? Explain that since the opposite sides are equal, the abscissa of D is b-a. To show the diagonals bisect each other, we show that they have the same midpoint. Mx $\overline{DB} = \frac{(b-a)+a}{2}, = \frac{b}{2}$, My $\overline{DB} = \frac{c+0}{2} = \frac{c}{2}$. Therefore the midpoint of DB is $(\frac{b}{2}, \frac{c}{2})$ The midpoint of AC is Mx $\frac{b+o}{2} = \frac{b}{2}$, My $= \frac{c+o}{2} = \frac{c}{2}$. The midpoint of $\overline{AC}$ is also $(\frac{b}{2}, \frac{c}{2})$. Therefore the diagonals bisect each other.





3. Prove that the line joining the midpoints of two sides of a triangle is equal to one-half the length of the third side. Find the midpoints of $\overline{AC}$ and $\overline{BC}$, Mx BC $= \frac{b+a}{2} = \frac{b}{2}$, My AC $= \frac{c+0}{2} = \frac{c}{2}$. The coordinates of the midpoint of AC or $(\frac{b}{2}, \frac{c}{2})$ = K, Mx $\overline{BC} = \frac{b+a}{2}$ My $= \frac{c+0}{2} = (\frac{b+a}{2}, \frac{c}{2})$ = L

The length of $\overline{AB}$ = a by counting. The length of $\overline{KL}$ is $\sqrt{(\frac{b}{2} - \frac{b+a}{2})^2 + (\frac{c}{2}-\frac{c}{2})^2} = \sqrt{(\frac{a}{2})^2} = \frac{a}{2}$. Therefore, the length of the segment joining the midpoints of two sides of a triangle = $\frac{1}{2}$ the length of the third side. Are they parallel?

Find the slopes. The slope $m\overline{AB} = \frac{0-0}{0-a} = 0$, the slope, $m\overline{KL} = \dfrac{\frac{c}{2}-\frac{c}{2}}{\frac{b}{2}-\frac{b+a}{2}} = 0$ therefore the lines are $\parallel$.

Hence the segment joining the midpoints of two sides of a $\triangle$ is $\parallel$ to the third side.

APPLICATIONS:

1. Show by using coordinate geometry that the segments joining the midpoints of any quadrilateral form a parallelogram.
2. Show by using coordinate geometry that the diagonals of a rhombus are perpendicular to each other.
3. Show by using coordinate geometry that the diagonals of a square are congruent.
4. Show by using coordinate geometry that one pair of opposite sides of a parallelogram are both parallel and congruent.

SUMMARY:

What factors do we take into consideration in deciding how to position a figure in the coordinate plane in order to prove general theorems?

HOMEWORK:

1. Show by using coordinate geometry that a median of an equilateral triangle is also an altitude.
2. Show by using coordinate geometry that the diagonals of a rectangle bisect each other.
3. Show by coordinate geometry that the diagonals of an isosceles trapezoid are congruent.

Lesson 17

AIM: How do we find the areas of geometric figures using coordinate geometry?

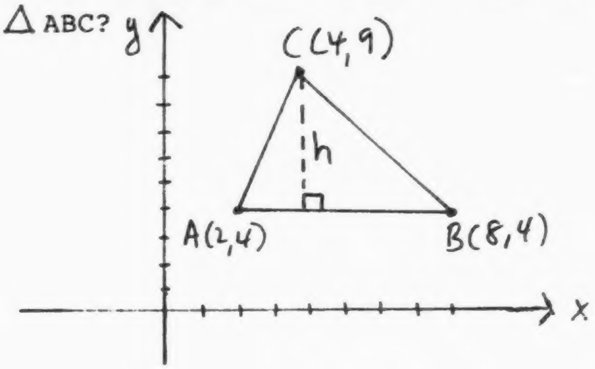
PERFORMANCE OBJECTIVES: The student will be able to...

1. Use formulas for the areas of triangles, parallelograms, rectangles, rhombuses, squares, and trapezoids to find the areas of these polygons when they are drawn in the coordinate plane.
2. Develop and apply additional formulas for finding the area of a rhombus ($A = \frac{1}{2}d_1d_2$) and the area of a square ($A = \frac{1}{2}d^2$)
3. Find the area of a triangle or quadrilateral having one or more sides parallel to either of the coordinate axes, by using the distance formula.

VOCABULARY: formula, rhombus, trapezoid

MOTIVATION:

1. Warm-up: Graph the triangle whose vertices are the points A(2,4), B(8,4) and C(4,9). How could we find the area of △ABC?

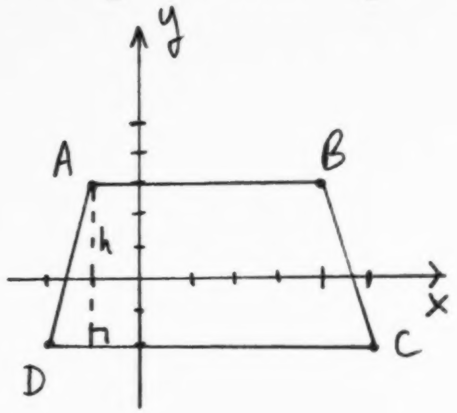


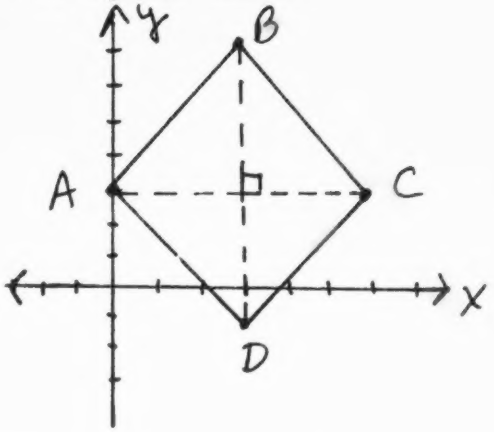
DEVELOPMENT:

1. Elicit the formula $A = \frac{1}{2}bh$, $b = 8-2 = 6$, $h' = 9-4 = 5$, $A = \frac{1}{2}.6.5$, $A = 15$
2. Find the area of a trapezoid ABCD with coordinates A(-1,3), B(4,3), C(5,-2), D(-2,-2).

Elicit the formula for area of a trapezoid as A trapezoid $= \frac{1}{2} h (b_1 + b_2)$.

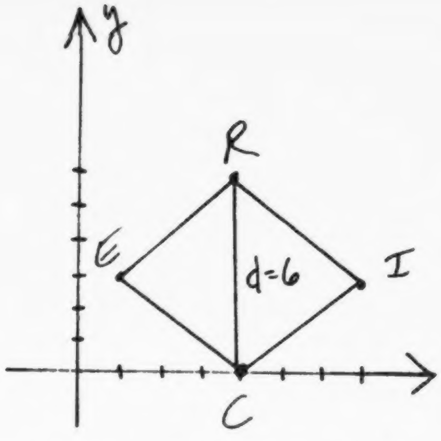
Elicit that the height = 5 units after it is drawn, base DC = 7 units, base AB = 5 units therefore the area $= \frac{1}{2} \cdot 5 (7+5) = \frac{1}{2} \cdot 60 = 30$ square units.





3. Consider rhombus ABCD with coordinates A(0,3), B(3,7), C(6,3) and D(3,-1). What is the area of this rhombus? The normal method of base times height would be difficult since it would be difficult to determine the height of the rhombus. Ask, "what other line segments would be helpful?" Elicit the diagonal $\overline{AC}$ would create two triangles whose area is easily found. What else will be needed to do this? Elicit the altitudes of $\triangle$ABC and of $\triangle$ADC. When both of these are drawn it should be clear that each is one-half $(\frac{1}{2})$ the other diagonal $\overline{BD}$. Why? (Diagonals of a p bisect each other). Therefore we get area of rhombus ABCD = area of $\triangle$ ABC + area of $\triangle$ ADC. That is, a rhombus $= \frac{1}{2}$ b.h $+ \frac{1}{2}$ b.h
If we call $\overline{AC}$ diagonal $1(d_1)$ and $\overline{BD}$ diagonal $2(d_2)$, we see that the area of a rhombus $= \frac{1}{2} . d_1 \cdot \frac{1}{2} d_2 + \frac{1}{2} d_1 \cdot \frac{1}{2} d_2$ multiplying we get $A = \frac{1}{4} d_1 . d_2 + \frac{1}{4} d_1 d_2$ and adding we get $A = \frac{1}{2} d_1 d_2$. Now we get $\overline{AC} = d = 6$, and $\overline{BD} = d_2 = 8$ so the area $= \frac{1}{2} . 6 . 8 = 24$ square units.
Note: The area of each $\triangle$ is equal to $\frac{1}{2} . 4 . 6 = 12$ and doubling this we see the area = 24. Hence we now have an alternate formula for the area of a rhombus which is equal to one-half the product of the diagonals
$(A = \frac{1}{2} d_1 . d_2)$

4. Consider quadrilateral ERIC with E(1,3), R(4,6), I(7,3) and C(4,0). Have the students plot the points and Ask, "What kind of a quadrilateral is ERIC?" "What is its area?" Elicit that it is a square. Without graphing, we could show that it is a square by using the distance and slope formulas.



To show that it is a square show that it contains a right angle. To find its area we could square the length of a side or use the rhombus formula for area. That is, $A = \frac{1}{2} d_1 d_2$. But what is true about d_1 and d_2 in a square? They are equal. Therefore, the formula can be simplified to $\frac{1}{2} d^2$ for the area of a square. By counting (or by using the distance formula) we see $d = 6$, therefore the area $= \frac{1}{2} \cdot 6^2 = \frac{1}{2}(36) = 18$ square units.

APPLICATIONS:

1. Find the area of a rectangle whose vertices are:
   (0,2), (8,2), (0,7), (8,7)
2. Find the area of a triangle if its vertices are as follows:
   a) (-2,0), (10,0), (0,8)
   b) (0,7), (0,-4), (4,4)
3. Find the area of a parallelogram whose vertices are:
   a) (1,0), (5,0), (3,3), (7,3)
   b) (-2,9), (-3,5), (5,9), (4,5)
4. Find the area of a trapezoid whose vertices are:
   a) (0,-2), (12,-2), (2,4), (7,4)
   b) (1,4), (1,-8), (4,1), (4,-4)
5. Find the area of a rhombus whose vertices are:
   a) (1,2), (4,7), (7,2), (4,-3)
   b) (0,2), (-3,4), (-6,2), (-3,0)
6. Find the area of a square whose vertices are:
   a) (0,2), (3,-1), (6,2), (3,5)
   b) (4,1), (0,-3), (-4,1), (0,5)

SUMMARY:

1. What is the formula for the area of a rhombus?
2. What is the formula for the area of a square using its diagonals.
3. If the length of each diagonal of a rhombus is doubled, how is the area affected?

HOMEWORK:

1. Find the area of a rectangle whose vertices are.
    a) (1,3), (9,3), (1,8), (9,8)
    b) (-1,-3), (6,-3), (6,1), (-1,1)
2. Find the area of a triangle whose vertices are.
    a) (0,-1), (12,-1), (2,7) b) (0,8), (0,-3), (4,5)
3. Find the area of a trapezoid whose vertices are.
    a) (-2,-3), (10,-3), (0,3), (5,3)
    b) (2,6), (2,-6), (5,3), (5,-2)
4. Find the area of a rhombus whose vertices are.
    a) (0,1), (3,6), (6,1), (3,4)
    b) (1,3), (-2,5), (-5,3), (-2,1)
5. Find the area of a square whose vertices are.
    a) (-1,1), (2,-2), (5,1), (2,4)
    b) (6,0), (2,-4), (-2,0), (2,4)

Lesson 18

AIM: How do we find the areas of geometric figures using coordinate geometry?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Find the area of a triangle or quadrilateral having no sides parallel to either of the coordinate axes, by enclosing the figure in a rectangle or by creating trapezoids.

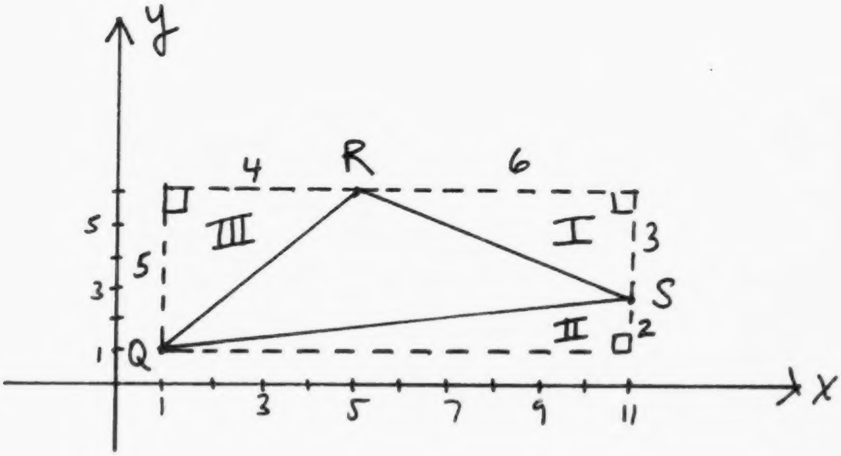
VOCABULARY: trapezoidal method, enclosing rectangles

MOTIVATION:

1. Warm-up: Plot the points Q(1,1), R(5,6), S(11,1). Find the area of $\triangle$QRS.

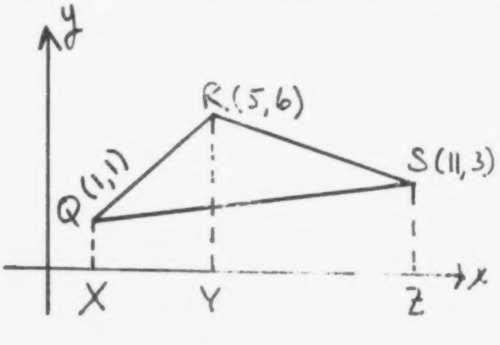
DEVELOPMENT:

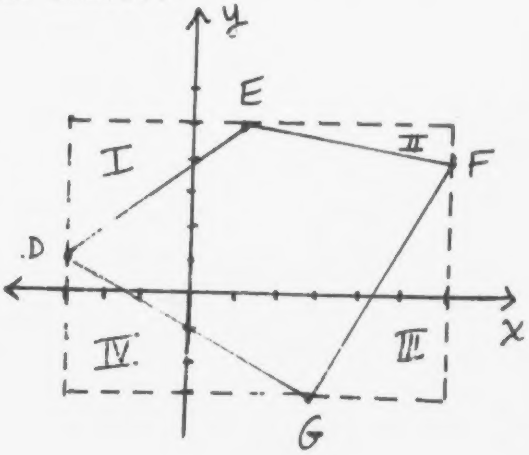
1. Plot the points Q, R, and S. Elicit since one side is parallel to a coordinate axis ($\overline{QS}$ // x-axis) we can apply the area formula by first drawing the altitude from R to $\overline{QS}$, intersecting $\overline{QS}$ at point H. Area $\triangle$ QRS $= \frac{1}{2}bh = \frac{1}{2}(QS)(RH) = \frac{1}{2}(10)(5) = 25$
   Suppose the coordinates of S are (11, 3).



Can we find the area of $\triangle$ QRS? Elicit, that we cannot easily find the length of an altitude. We can, however, create another figure which will allow us to find the area of $\triangle$ QRS indirectly. By enclosing $\triangle$ QRS in a rectangle we can find the area of the rectangle and subtract the areas of the right triangles that are formed. That is, the area of $\triangle$ QRS = area of the rectangle -(area $\triangle$ I + area $\triangle$ II + area of $\triangle$ III). Elicit, A rectangle = b·h = 10x5 = 50, A $\triangle$ I = $\frac{1}{2}\cdot 6\cdot 3 = 9$, A $\triangle$ II = $\frac{1}{2}\cdot 10\cdot 2 = 10$, A $\triangle$ III = $\frac{1}{2}\cdot 4\cdot 5 = 10$. Therefore, area $\triangle$ QRS = 50 - (9+10+10) = 50-29 = 21 square units.

2. Another method involves creating trapezoids. Let us use this method with the same $\triangle$ QRS. Q(1,1), R(5,6), and S(11,3),
   The area of trapezoid XYRQ = $\frac{1}{2}\cdot 4(1+6) = 14$.
   The area of trapezoid YXSR = $\frac{1}{2}\cdot 6(3+6) = 27$.
   The area of trapezoid XQRZ (shaded area) = $\frac{1}{2}\cdot 10(1+3) = 20$. The area of $\triangle$QRS = A trapezoid XYRQ + A trapezoid YZRS -A trapezoid XQSZ. The area $\triangle$ QRS = 14+27-20 = 21 square units. This is the same answer.





3. Let's find the area of quadrilateral DEFG with D(-3,1), E(1,5), F(6,4), G(3,-3).
   Elicit a medial summary: What are the two ways of doing this? Circumscribe a rectangle around it or create trapezoids. For this figure it appears that the rectangle method is best.
   Note: The trapezoid method can also be used.

Area of rectangle = b.h = 9.8 = 72 $A\triangle I = \frac{1}{2}.4.3 = 6$, $A\triangle II = \frac{1}{2}.1.5 = 2\frac{1}{2}$, $A\triangle III = \frac{1}{2}.3.7 = 11\frac{1}{2}$, $A\triangle IV = \frac{1}{2}.6.3 = 9$. Therefore the area of DEFG = $72 - (6+2\frac{1}{2}+11\frac{1}{2}+9) = 72-29 = 43$ square units.

APPLICATIONS:

1. Find the area of each of the following triangles whose vertices are:
   a) (2,6), (8,4), (5,-1) c) (-7,5), (3,2), (-3,-4)
   b) (-4,3), (4,9), (11,-2) d) (-4,-5), (2,7), (6,-1)
2. Find the area of the quadrilateral whose vertices are:
   a) (1,7), (3,4), (9,5), (5,11)
   b) (2,-3), (4,2), (-1,6), (-7,-1)
3. Find the area of polygon ABCDE when the vertices are: A(-4,-2), B(3,-5), C(7,-1), D(4,6) and E(-2,3).
4. On the same set of axes draw the lines indicated and find the area of the quadrilateral they form.
   $Y = X$, $Y = -X$, $X = 7$, $Y = -3$

SUMMARY:

What are the two methods for finding the area of a polygon in coordinate geometry?

HOMEWORK:

1. Find the area of each of the following triangles whose vertices are:
   a) (1,5), (4,1), (-3,-4) b) (-3,6), (1,4), (5,-3)
2. Find the area of the quadrilateral whose vertices are:
   a) (7,1), (4,3), (5,9), (11,5)
   b) (2,-6), (4,-1), (-1,3), (-6,-4)
3. Find the area of the polygon whose vertices are:
   (7,1), (3,-3), (-2,-2), (-5,2), (0,8) and (6,5)
4. On the same set of axes draw $Y = 2X$, $X = 4$, $Y = -6$, and $X+Y = -5$. Find the area of the quadrilateral formed.

Lesson 19

AIM: What is the meaning of locus and how do we discover a probable locus? (Part I: The 5 Fundamental Loci)

PERFORMANCE OBJECTIVES: The student will be able to ...

1. State the definition of locus and that the plural of locus is loci.
2. State the five fundamental loci.
3. State and apply a procedure for discovering a probable locus.

VOCABULARY: locus, loci, equidistant

MOTIVATION:

Draw a point on the chalkboard and label it point P. Select a student to come to the chalkboard. Ask the student to use a yardstick to locate points which are at a distance of 18 inches from point P. After locating a representative sampling of such points ask the class, "How would you describe the set of all points that are a distance of 18 inches from point P?"

DEVELOPMENT:

Introduce the term locus (plural: loci) and define it as the set of all points, and only those points, which satisfy one or more given conditions. Using the motivation problem, introduce a systematic procedure to help in determining a probable locus. For example,

Step 1: Draw a diagram which includes the given information (in our example, draw point P) and several representative points which satisfy the stated condition (in our example, that the points be 18 inches from point P).

Step 2: Continue drawing points until a pattern emerges.

Step 3: Connect the points using a broken line or curve.

Step 4: Describe the locus in a complete and grammatically correct sentence.

Step 5: Verify by checking that:
(a) if a point is on the locus, then it satisfies the given condition(s), and
(b) if a point satisfies the given condition(s), then the point lies on the locus.

NOTE: Relate step 5 to the requirements of the definition of locus. In better classes, explain why the definition of locus requires proving the conditional statement and its converse to prove a locus is correct or proving the statement and its inverse.

Using the motivation problem, elicit the statement of the first of the fundamental loci:

The locus of all points at a fixed distance from a given point is a circle with the given point as the center and the fixed distance as the length of its radius.

On a reserved section of the chalkboard begin listing using a table format the fundamental loci to be introduced during this period:

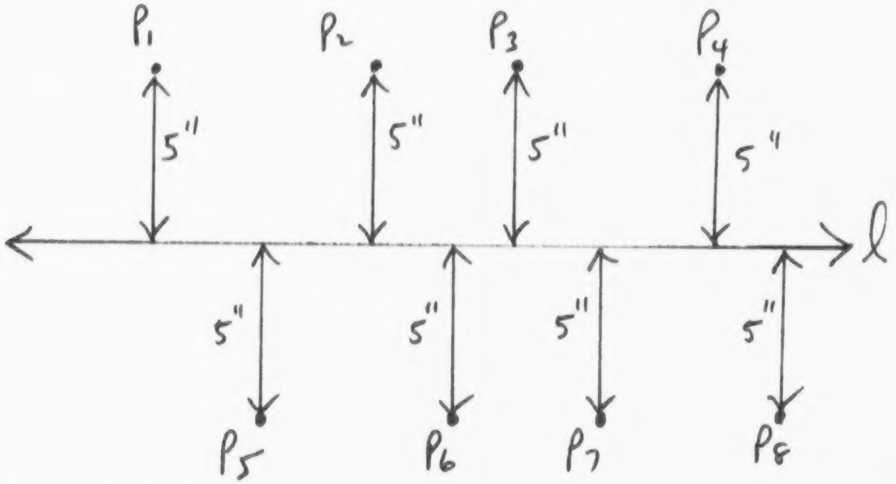
Conditions	Locus
1. Given a point and a fixed distance, find points that are at the fixed distance from the given point.	1. The locus of points is a circle having the given point as its center and the fixed distance as the length of its radius.

The first of the fundamental loci deals with finding the set of all points at a given distance from a given point, and is derived from the definition of a circle.

Ask the class to apply the previously developed five step procedure to try to discover the locus of all points at a given distance (for example, 5 inches) from a given line.

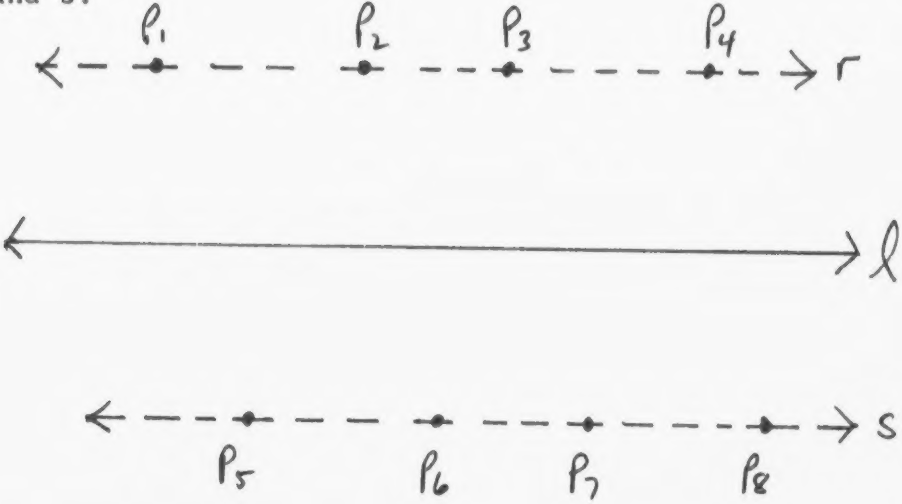
Elicit:

Step 1:



P_1
P_2
P_3
P_4
5"
5"
5"
5"
l
5"
5"
5"
5"
P_5
P_6
P_7
P_8

Step 2 and 3:



P_1
P_2
P_3
P_4
r
l
s
P_5
P_6
P_7
P_8

Step 4:

The locus is a pair of lines parallel to the given line 1 and each at a distance of 5 inches from the given line 1.

Step 5:

Every point on lines r and s satisfy the locus condition and every point that satisfies the locus condition lies on either line r or line s.

Add to the CONDITIONS column of the table:

2. Given a line and a distance, find points that are at the given distance from the line.

Add to the LOCUS column of the table:

2. The locus of all points at a given distance from a given line is a pair of lines, each parallel to the given line and at the given distance from it.

Advise the class that we would now like to investigate the locus of points that are EQUIDISTANT from two points or two lines.

Using a similar procedure, draw illustrative diagrams and elicit the statements of the remaining 3 fundamental loci:

- The locus of all points equidistant from two given points is the perpendicular bisector of the line segment joining the two points.
- The locus of all points equidistant from two parallel lines is a third line parallel to the two lines and midway between them.
- The locus of all points equidistant from two intersecting lines is the pair of lines which bisect the angles formed by the two given lines.

These should also be summarized by adding to the table.

APPLICATIONS:

1. Have students draw an appropriate diagram and then identify the locus principle which may be used to describe:
    a. the path of a bowling ball rolling straight down an alley to the kingpin.

b. the boundary of the path a dog tied to a stake with a rope 6 feet long may follow assuming the dog keeps the rope taut.
c. the path of a train that runs on a straight track so that it is always the same distance away from two cities.
d. the path of a child who walks so that he is always the same distance away from two intersecting roads.
e. the path of hikers that are always the same distance away from a stream that doesn't turn or twist.

2. What is the locus of all points that are a distance of less than 3 inches from point P?

3. The length of a side of square ABCD is 10. Describe fully the locus of points:
   a. equidistant form vertex A and B.
   b. equidistant from vertex A and C.
   c. 6 units from side AB.
   d. equidistant from sides AD and BC.
   e. that are in the interior of the square and which are 5 units from vertex A.
   f. equidistant from sides AB and BC.

4. If a coin rolls along a flat surface in a plane, what is the locus of the center of the coin?

5. What is the locus of the centers of all circles that have congruent radii and which passes through a fixed point?

SUMMARY:

1. Describe the five step procedure that can be used to discover a probable locus.

2. Louis stated that the locus of points 2 inches from a given line m is a line s parallel to line m and 2 inches from it since every point on line s is 2 inches from line m. What error in reasoning did Louis make?

HOMEWORK:

In each of the following draw an appropriate diagram and descrive fully the locus.

1. The locus of points:
   a. 4 cm. from point P
   b. less than 4 cm. from point P
   c. more than 4 cm. from point p
2. AB = 10. The locus of points equidistant from the endpoints of $\overline{AB}$.
3. In rectangle MATH, AM = 8 and AT = 12. The locus of points:
   a. equidistant from $\overline{MH}$ and $\overline{AT}$.
   b. 7 units from $\overline{AM}$.
4. Given right $\triangle$ ABC where BC = 3, AC = 4, and AB = 5, the locus of points:
   a. 1 unit from vertex C.
   b. 2 units from side AC.
   c. equidistant from vertex A and B.
   d. equidistant from the sides of angle BAC.
   e. in the interior of the triangle and 3 units from side BC.
5. A dime is rolled around a half dollar. The locus of the center of the dime.
6. The locus of the midpoints of the radii of a given circle.
7. The locus of the centers of the circles that have congruent radii and pass through a fixed point.
8. The length of a side of a square is two inches. The locus of all points in the interior of the square and 1/4 inch from a side?

Lesson 20

AIM: What is the meaning of locus and how do we discover a probable locus? (Part II: Geometric Applications)

PERFORMANCE OBJECTIVES: The student will be able to ...

1. State the definition of locus.
2. State the five fundamental loci and apply them in discovering a probable locus in a variety of geometric situations.
3. Prove geometrically (or describe how to prove) that the locus of points -
    a. equidistant from the endpoints of a line segment is the perpendicular bisector of the segment.
    b. equidistant from the sides of an angle is the ray that bisects the angle.

VOCABULARY: locus, loci, equidistant, concentric circle

MOTIVATION:

Find the locus of the vertices of all isosceles triangles that can be drawn using a given line segment as the base.

DEVELOPMENT:

Review the motivation problem and elicit that the vertex of an isosceles triangle is a point that has the property that it is equidistant from the endpoints of the base. After reviewing the five fundamental loci, elicit that the third of these applies. The locus of the vertices of all isosceles triangles that can be drawn using a given line segment as the base is the line that is the perpendicular bisector of the base.

Using their knowledge of geometry, challenge students to prove that the locus of all points equidistant from two given points (that is, the endpoints of a segment) is the perpendicular bisector of the segment joining the two points. Stress that the proof consists of two distinct parts.

Part I. To show that any point equidistant from the endpoints of the segment is on the perpendicular bisector of the segment.

Part II. To show that any point on the perpendicular bisector of the segment is equidistant from the endpoints of the segment.

Outline of proof of Part I

Given: $\overline{AB}$ with point C such that CA = CB.

Prove: Point C lies on the perpendicular bisector of $\overline{AB}$.

Let D be the midpoint of $\overline{AB}$. Draw the line determined by points C and D. Show that $\triangle ADC \cong \triangle BCD$ by SSS $\cong$ SSS. Hence $\angle CDA \cong \angle CDB$, therefore line CD is perpendicular to $\overline{AB}$.

Outline of proof of Part II

Given: D is the midpoint of $\overline{AB}$ and $\overline{CD}$ is the perpendicular bisector of $\overline{AB}$. P is any point on line CD.

Prove: PA = PB

Show right $\triangle ADP \cong$ right $\triangle BDP$ by SAS $\cong$ SAS. Hence PA = PB.

APPLICATIONS:

1. What is the locus of all points equidistant from the bases of an isosceles trapezpoid?
2. Find the locus of all points in the interior of a circle which are equidistant from the endpoints of a given chord.

3. Find the locus of the centers of the circles which are tangent to each of two parallel lines.
4. Concentric circles are circles having the same center. Find the locus of all points more than 3 inches and less than 7 inches from a given point.
5. Find the locus of the centers of all circles tangent to two concentric circles having a radius of 3 inches and a radius of 7 inches.
6. Find the locus of the points of contact of the set of tangents drawn from a fixed point to a set of concentric circles.
7. Describe fully the locus of all points that are equidistant from the vertices of a right triangle.
8. What is the locus of the centers of the circles having congruent radii and tangent to a given line?
9. What is the locus of the centers of all circles tangent to two intersecting lines?
10. Find the locus of the vertices of a triangle having a fixed base 10 inches in length and whose median drawn to that base has a length of 5 inches.

SUMMARY:

1. Why does a locus proof consist of two parts?
2. What two conditions are necessary for proving that the locus of all points equidistant from the sides of an angle is the ray that bisects the angle? (The proof will be included in the homework assignment).

HOMEWORK:

1. Find the locus of points equidistant from sides AD and BC of parallelogram ABCD.
2. What is the locus of a point on the rim of a wheel turning on a stationary axle?

3. What is the locus of the midpoints of line seqments whose endpoints lie on a given pair of parallel lines?

4. ABCD is a quadrilateral. The locus of points equidistant from $\overline{AB}$ and $\overline{AD}$ must include point C if ABCD is a :

   a) trapezoid b) rectangle c) parallelogram
   d) rhombus

5. Triangle ABC has an area of 40 square units. If AB = 10 and is fixed in position, then the locus of point C is two lines parallel to $\overline{AB}$ and x units from $\overline{AB}$. The value of x is:

   a) 4 b) 6 c) 8 d) 10

6. What is the locus of points that are always a fixed distance from a given circle?

7. What is the locus of the centers of circles that have a given line segment as a common chord?

8. What is the locus of the midpoints of line segments whose endpoints are one of the vertices of a triangle and any point on the side opposite that vertex?

9. What is the locus of the midpoints of line segments whose endpoints lie on a given pair of parallel lines?

10. What is the locus of the centers of circles that are tangent to both sides of a given angle?

11. Prove that the locus of all points equidistant from the sides of an angle is the ray that bisects the angle.

12. Prove that the locus of all points at a given distance from a given line is a pair of lines, each parallel to the given line and at the given distance from it.

13. Prove that the locus of all points equidistant from two parallel lines is a third line parallel to the two lines and midway between them.

Lesson 21

AIM: What is the meaning of locus and how do we discover a probable locus? (Part III: Finding a Locus by Construction)

PERFORMANCE OBJECTIVES: The students will be able to ...

1. State the five fundamental loci and relate them to basic constructions involving bisectors and parallel lines.
2. Construct (or describe how to construct) the locus of points at a given distance from a given point, at a given distance from a given line, equidistant from the endpoints of a segment, equidistant from two parallel lines, and equidistant from the sides of an angle.

VOCABULARY: no new terms

MOTIVATION:

Tell students to draw a line segment of any convenient length in their notebooks. Ask, "How could you find the exact location of the locus of all points that are equidistant from the endpoints of the segment drawn in your notebooks?"

DEVELOPMENT:

In considering the motivation, elicit the need to construct the perpendicular bisector of the segment. After reviewing the required construction, discuss the relationship between the fundamental loci and basic constructions related to bisectors and parallel lines.

Next, ask students to construct the locus of all points:

1. equidistant from the sides of acute angle ABC.

2. a given distance d from a line. (Have students construct a perpendicular at any point P on the line and then to mark off a length equal to d on this perpendicular on either side of the line. Call these points A and B so that AP = BP = d. Through points A and B construct lines parallel to the given line.)

APPLICATIONS:

1. Given distances a and b (b > a). Find by construction the locus of all points more than a units but less than b units from a given point P.
2. Given parallelogram ABCD. Construct the locus of all points equidistant from $\overline{AB}$ and $\overline{CD}$.

SUMMARY:

Which locus theorem is related to the construction of:

a. a perpendicular bisector?

b. an angle bisector?

HOMEWORK:

1. Given obtuse < ABC where B is an obtuse angle, construct the locus of points:
    a. equidistant from points A and C.
    b. equidistant from $\overline{AB}$ and $\overline{BC}$.
2. Construct the locus of points equidistant from the bases of a trapezoid.
3. Draw a line segment having any convenient length (say t units in length). Construct the locus of points:
    a. less than t units from a given point p.
    b. t units from a given line l.

LESSON 22

AIM: How do we find the points in a plane which satisfy two different sets of conditions?

PERFORMANCE OBJECTIVES: The students will be able to ....

1. Draw the locus of points satisfying the first of two conditions, the locus of points satisfying the second condition, and locate the points of intersection of these loci.
2. Determine that the locus of points satisfying each condition may intersect in no points, in a single point, in two points, or in more than two points.
3. Determine the intersection of loci by sketching on blank paper when given general conditions involving numbers.

VOCABULARY: no new terms

MOTIVATION:

Locate all points equally distant from two points A and B and one-half inch from a third point C.

DEVELOPMENT:

1. Through discussion of the motivation problem, elicit the need for finding each locus separately and then finding the intersection of the loci. Develop with students the following plan:

   (a) Find the locus of points equidistant from points A and B to be the perpendicular bisector of the segment AB, call it $\overleftrightarrow{FG}$. Draw $\overleftrightarrow{FG}$ using color chalk.

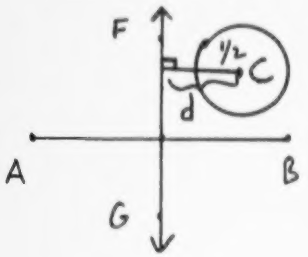
   (b) Find the locus of points one-half inch from point C to be the circle with center C and radius 1/2 inch. Draw the circle with colored chalk.

(c) Find the locus of points which satisfy both conditions by drawing the loci on the same diagram and finding the number of points of intersections.

Elicit that the position of C with respect to $\overleftrightarrow{FG}$, the perpendicular bisector of $\overline{AB}$, determines the number of points of intersection as follows:

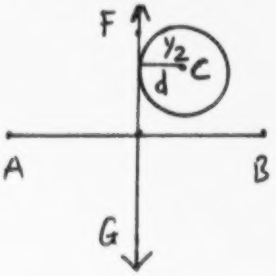
Distance from C to $\overleftrightarrow{FG}$ = d

Case I: d>1/2



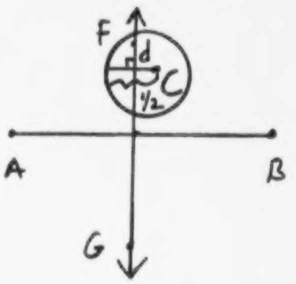
Ans. no points

Case II: d=1/2



Ans. 1 point

Case III: d<1/2



Ans. 2 points

2. Pose a second problem for class discussion. Ask: "If $\overleftrightarrow{LM}$ and $\overleftrightarrow{RS}$ are two parallel lines 10 inches apart and A is a point on $\overleftrightarrow{LM}$, how many points are there which are equidistant from $\overleftrightarrow{LM}$ and $\overleftrightarrow{RS}$ and 7 inches from point A?"

Elicit from students the following method of solution:

1. Draw the given point and lines labeling the diagram with the given conditions.
2. Find the first locus, draw it in color. The locus of points equidistant from two parallel lines 10 inches apart is a third parallel line midway between them, i.e. 5 inches from each line.
3. Find the second locus, draw it in a different color. The locus of points 7 inches from a given point is a circle with the given point A as center and the radius of 7 inches.

4. Find the number of points of intersection of the two loci. Label them C and D. Write the solution set is two points.

APPLICATIONS:

1. Points R and S are 6 centimeters apart. What is the locus of points equidistant from R and S and
   (a) 4 cm. from R?
   (b) 2 cm. from S?
   (c) 3 cm. from R?

2. Two straight lines PQ and RS intersect at T. How many points are 3 units from T and also equidistant from $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$?

3. A point P is 1 unit from line AB. Locate the points which are 3 units from line AB and 2 units from P.

4. Given acute triangle ABC, what is the locus of points equidistant from $\overline{AB}$ and $\overline{AC}$ and also equidistant from vertices A and B?

5. Given two concentric circles whose center is O and the lengths of whose radii are 4 and 10, and given a straight line AB through O
   (a) Describe fully the locus of points equidistant from the two concentric circles.
   (b) Describe fully the locus of points at a given distance d from line AB.
   (c) Find the number of points that satisfy the conditions in both part (a) and (b) if
      (1) d=4
      (2) d=8
      (3) d=10

SUMMARY:

1. Write a procedure which can be used to find the points in the plane which satisfy two different sets of conditions.

2. Use the method described in exercise 1 to find all points equally distant from three given lines if
   - (a) the lines intersect in 3 points
   - (b) two of the lines are parallel and the third line is a transversal
   - (c) all three lines are parallel.

HOMEWORK:

1. How many points are there which are 3 inches from a given line and 2 inches from a given point in that line?

2. What are the number of points which are equidistant from two given parallel lines and at the same time equidistant from two given points on one of these lines?

3. How many points are there that are equidistant from two given points R and S also 3 inches from the straight line through R and S?

4. Point P is 2 inches from given line AB. The number of points in $\overleftrightarrow{AB}$ that are 6 inches from P is
   (a) 0 (b) 1 (c) 2

5. Line segment PQ is 2 inches long. The number of points 2 inches from P and 2 inches from Q is
   (a) 0 (b) 1 (c) 2

6. Two points A and B are 9 inches apart. How many pointgs awre there which are 14 inches from A and 5 inches from B?

7. Two points A and B are 6 units apart. How many points are equidistant from A and B and also 4 units from B?

8. In triangle RST, the point that is equidistant from the three vertices R, S, and T is the intersection of (a) the angle bisectors (b) the perpendicular bisectors of the sides (c) the medians (d) none of the first three choices.

9. Given line RS and a point T located 10 units from $\overline{RS}$.
   (a) Describe fully the locus of points at a distance of 4 units from $\overleftrightarrow{RS}$.
   (b) Describe fully the locus of points at a distance of d units from point T.
   (c) How many points will satisfy the conditions of both (a) and (b) if
      (1) d = 5
      (2) d = 7

10.Given line AB and a point P that is four units from $\overleftrightarrow{AB}$. (a) Sketch the locus of points equidistant from point P and $\overline{AB}$. (b) Sketch the locus of points 5 units from $\overleftrightarrow{AB}$. (c) How many points satisfy the conditions described in parts (a) and (b) simultaneously?

Lesson 23

AIM: What is the meaning of "equation of a locus"?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Explain the meaning of the phrase "equation of a locus."
2. Use coordinate geometry to write the equation of the locus of points whose abscissas are constant and the equation of the locus of points whose ordinates are constant.
3. Write the equation of the locus of points satisfying given conditions.
4. Graph the locus of points described by given conditions.
5. Determine whether a given point is on a locus whose equation is given.

VOCABULARY: equation of a locus, abscissa, ordinate

MOTIVATION:

On a pair of coordinate axes, sketch and describe the locus of points whose abscissas are four.

DEVELOPMENT:

1. Ask: What is the equation of the line found in answer to the motivation problem? Elicit the equation of the line whose abscissas are four is $x=4$.
   Ask: Do all points on the line satisfy the equation? Does any point not on the line satisfy the equation?

   Explain that the equation $x=4$ is called the equation of the locus because
   (1) every point on the locus satisfies the equation
   (2) all points not on the locus do not satisfy the equation.

   Discuss the locus of points whose abscissas are -3 and its equation. Elicit that the locus of points whose abscissas are the constant a is the line that is parallel to the y axis and whose equation is $x=a$.

2. Ask: What is the locus of points whose ordinates are 5? What is the equation of this locus?

   Have students generalize that the locus of points whose ordinates are the constant b is the line that is parallel to the x axis and whose equation is y=b.

3. Direct students to graph the line determined by the points (0,1), (2,5), (3,7), (1,3) and (-1,-1). Ask: Describe the relation which exists between the abscissas and ordinates for all points on this line.

   Elicit that the ordinate is one more than twice the abscissa. Explain that the line is the locus of points whose ordinates are one more than twice their abscissas and the equation of the locus is y=2x+1.

   Ask students to write the equation of the locus of points in which:

   (a) the ordinate is 2 more than four times the abscissa
   (b) the abscissa is 3 less than twice the ordinate
   (c) the abscissa and ordinate have a sum of 6

4. Direct students to write the equation of the locus of points whose abscissas are 5 less than their ordinates. Ask: How can we determine if a given point lies on this locus? Elicit the points must satisfy the equation. Consider the points (5,10), (-5,0), (0,5) and (0,-5).

APPLICATIONS:

1. In each of the following write the equation of the locus of points which satisfy the given condition and then draw its graph.
   (a) points whose abscissas are -7
   (b) points whose ordinates are 0
   (c) points in which the ordinate is 1/2 its abscissa

(d) points in which the abscissa is 3 times the ordinate

(e) points in which the abscissa is 5 less than 4 times the ordinate

(f) points whose ordinates are 3 units from the x axis and below it

(g) points passing through the point with coodinates (3,5) and perpendicular to the x-axis

(h) points passing through the point with coordinates (-1,7) and parallel to the x-axis

(i) points 3 units to the right of the line with equation x=2

(j) points 5 units below the line with equation y=3

2. Determine which of the points to the right of each line are on the locus whose equation is given.

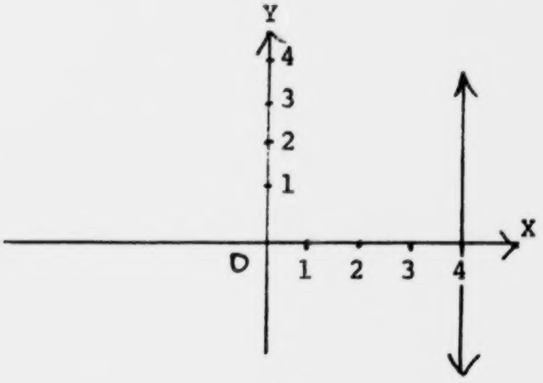
(a)	2x+y=9	(0,9)	(9,0)	(1,6)
(b)	x=12	(8,4)	(0,12)	(12,5)
(c)	y=1/3 x	(7,21)	(-9,-3)	(0,0)
(d)	y=3x-5	(4,7)	(0,-5)	(1,2)

SUMMARY:

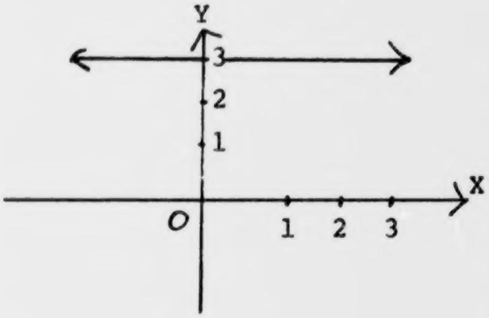
For each graph below

(a) describe the given set of points in terms of locus conditions which they satisfy

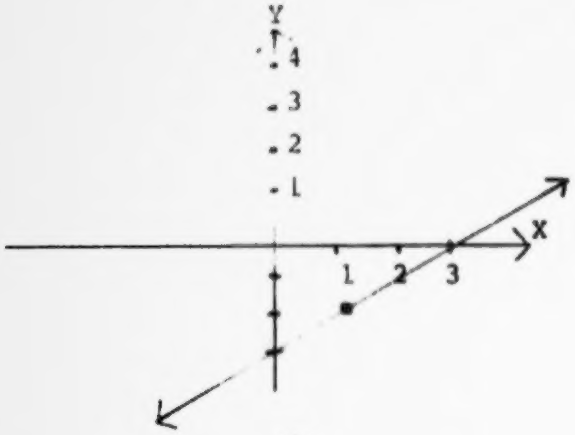
(b) write the equation of the locus



(1)



(2)



(3)

HOMEWORK:

1. Determine whether the given point is on the locus whose equation is given.
   - (a) $x-y=6$ $(12,-6)$
   - (b) $3x+y=8$ $(2,2)$
   - (c) $y+x=12$ $(14,-2)$
   - (d) $x=-5$ $(-5,5)$
   - (e) $y=7$ $(7,2)$

2. Find k so that the graph of the given equation will pass through the given point.
   - (a) $x-2y=k$ $(3,4)$
   - (b) $5x+2y=k$ $(-2,3)$

3. A point is on the locus whose equation is given. Find its abscissa if its ordinate is the value indicated.
   - (a) $x+y=15$ $y=3$
   - (b) $3x-2y=8$ $y=-2$
   - (c) $2x+y=30$ $y=3x$

4. Write the equation of the locus and graph the locus
   - (a) whose abscissa is 10
   - (b) whose ordinate is -9
   - (c) whose abscissa is 7 times its ordinate
   - (d) whose abscissa is 3 more than twice its ordinate
   - (e) whose ordinate is 5 less than three times it abscissa
   - (f) the sum of whose coordinates is 15
   - (g) whose ordinate decreased by its abscissa is -2
   - (h) which passes through the point (-3,1) and is perpendicular to the y axis

Lesson 24

AIM: How do we find the equation of the locus of points at a given distance from a given point?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Describe the locus of points at a given distance from a given point as a circle, with given point as its center and the given distance as its radius.
2. Use the distance formula to derive the equation of a circle whose center is the origin and whose radius has a length r.
3. Use the distance formula to derive the general equation of a circle whose center is any given point in the plane and whose radius has a length r.
4. Write the equation of a circle, given the coordinates of its center and the length of its radius.
5. Find the coordinates of the center and the length of the radius of a circle whose equation is given in the form $(x-h)^2+(y-k)^2=r^2$.
6. Write an equation of a locus of points whose distance from a specific point is given and determine whether any other point whose coordinates are given is on the locus.

VOCABULARY: No new terms.

MOTIVATION:

On a pair of coordinate axes sketch the locus of points five units from the origin. Describe the locus. What is the equation of the locus?

DEVELOPMENT:

1. Elicit from students that the locus sketched is a circle with center at the origin and radius 5. Review fundamental locus theorem 4: "The locus of points at a given distance from a given point is a circle with the given point as center and the given distance as the radius."

   Ask: How did we define equation of a locus in the previous lesson? Discussion will lead students to conclude that every point on the locus must satisfy the equation of the locus . Have students find the coordinates of points which are obviously on the locus using the diagram drawn in answer to the motivation probelm. Make a table of these valules as follows:

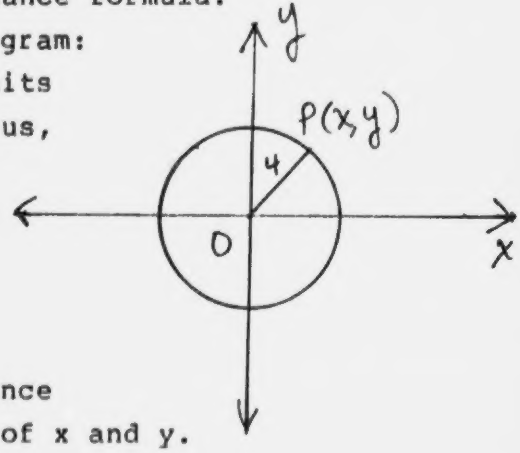
x	5	0	-5	0
y	0	5	0	-5

   Ask: How can we determine if a specific point, for example (3,4), lies on this circle?
   Elicit any point on the circle must be a distance of 5 units from the origin. Therefore we can use the distance formula to test the points as follows.
   Test the points A(3,4) , B(-4,3) , C(2,3) using

   $$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

   Add the points which satisfy the distance requirement to the table made earlier. Then ask: "Can you now find an equation in x and y which all the points found will satisfy?"

Explain that there is a *more direct way of finding* the equation of the locus of points at a given distance from the origin using the distance formula. Ask using the accompanying diagram: "If the given distance is 4 units and P is any point on this locus, what must be true of P?" Elicit OP=4. Explain since P represents any point, its coordinates are (x,y). Have students use the distance formula to represent the distance from the origin to P in terms of x and y.



Derive the equation $x^2+y^2=16$.

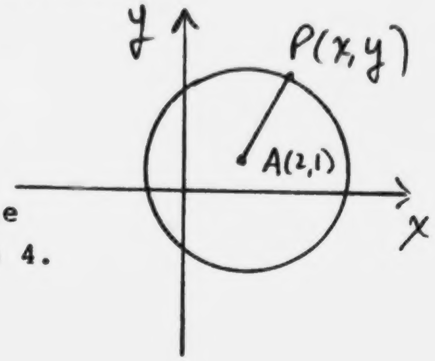
Ask: If the radius were r units and the center were the origin, how would the derivation differ?

Elicit the equation of the circle whose center is the origin and whose radius has length r is $x^2+y^2=r^2$. Do applications from 1 to 5.

2. Ask: How can we find the equation of the circle whose center is at A(2,1) and whose radius has a length of 4 units?

Elicit the method is exactly the same as that used in the previous exercise.

Sketch the graph of the locus and derive the equation of the new circle with the class as follows.



AP = 4

$(x-2)^2 + (y-1)^2 = 4^2$

is the equation of the circle with center (2,1) and radius 4.

$(x-2)^2 + (y-1)^2 = 16$

Ask: What is the equation of the circle whose center is at C(h,k) and whose radius has length $\underline{r}$?

Derive the equation of the circle described. Have students state the general equation of the circle with center (h,k) and radius r is

$(x-h)^2 + (y-k)^2 = r^2$

Do applications from 6 through 9.

APPLICATIONS:

1. Write the equation of the circle given the center C and the radius r if

    (a) C(0,0) and r= 6
    (b) C(5,1) and r=3
    (c) C(-5,0) and r=4
    (d) C(-4,-2) and r=9/2
    (e) C(-3,-3) and r=2/5

2. Write the equation of the locus of points 9 units from A(7,0).

3. Write the equation of the locus of points whose distance from the origin is 3/4.

4. Find the center and radius of the given circle.

   (a) $x^2y^2 = 36$
   (b) $x^2=y^2 = 5$
   (c) $2x^22y^2 = 10$

5. Determine if the following points lie on the locus whose equation is $x^2+y^2=50$.

   (a) (-5,5) (b) (5-,0) (c) (-7,-1)

6. What is the equation of the circle given the center C and the radius r?

   (a) C (5,1) r=13
   (b) C (-3,2) r=10
   (c) C (3,-5) r= 7

7. What is the center and radius of each circle?

   (a) $(x-3)^2 + (y-5)^2 = 49$
   (b) $(x+2)^2 + (y+6)^2 =121$

8. Write an equation of the circle with center (4,-1) and which passes through the point with coordinates (5,2).

9. Find the radius and the coordinates of the center of each circle.

   (a) $x^2 +y^2 = 100$ (d) $(x+3)^2 + (y-4)^2 = 4$
   (b) $4x^2 +4y^2 = 400$ (e) $(x-5)^2 +y^2 = 16$
   (c) $2x^2 +2y^2 =20$

SUMMARY:

1. In each of the following determine if the given equation is the equation of a circle. If it is circle, find the coordinates of the center and the length of the radius. If it is not a circle, give a reason for your answer.

   (a) $x^2 + y^2 = 1$
   (b) $2x^2 + 2y^2 = 10$
   (c) $x^2 - y^2 = 1$
   (d) $(x+5)^2 + (y-5)^2 = 16$
   (e) $x + y = 9$
   (f) $x^2 + y^2 = -4$
   (g) $x^2 + y^2 = 0$

HOMEWORK:

1. Write the equation of the circle with center at the origin and radius r if

   (a) $r=6$ (b) $r=1/3$ (c) $r=4/3$

2 Write an equation of a circle with center C and radius r if

   (a) C $(0,-3)$ and $r+1$
   (b) C $(5,-7)$ and $r=2/3$

3. Find the radius and the coordinates of the center of the circle whose equation is given. Sketch the graph

   (a) $x^2 +y^2 = 49$
   (b) $4x^2 + 4y^2 = 25$
   (c) $(x-6)^2 + (y+3)^2 = 16$
   (d) $(x-2)^2 + (y-4)^2 = 18$
   (e) $16(x+4)^2 + 16(y+5)^2 = 64$

4. Find the equation of the circle with center $C(-2,-2)$ and radius of length 5. Determine if the point $P(0,-1)$ lies on the circle.

5. Find the equation of the circle
   (a) whose center is at the orgin and passes through the point with coordinates $(8,0)$
   (b) whose center is $(4,6)$ and passes through the point having coordinates $(4,10)$

6. Show that the point with coordinates $(2,-5)$ lies on the circle whose equation is $(x-3)^2 + (y+4)^2 = 2$.

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LESSON 25

AIM: How do we find the locus of points equidistant from three given noncollinear points?

PERFORMANCE OBJECTIVES: The student will be able to ....

1. State that the locus of points equidistant from two given points as the perpendicular bisector of the line segment joining the points.
2. Write the equation of the locus of points equidistant from any two given points in the plane, using the midpoint formulas and the slopes of perpendicular lines.
3. Draw the locus of points equidistant from two given points.
4. Determine that the locus of points equidistant from three given noncollinear points is a single point, which is the center of the circle circumscribing the triangle whose vertices are the three given points.
5. Construct a circle circumscribed about a given triangle.

VOCABULARY: inscribed polygon, circumscribed circle, review perpendicular bisector of a segment.

MOTIVATION:

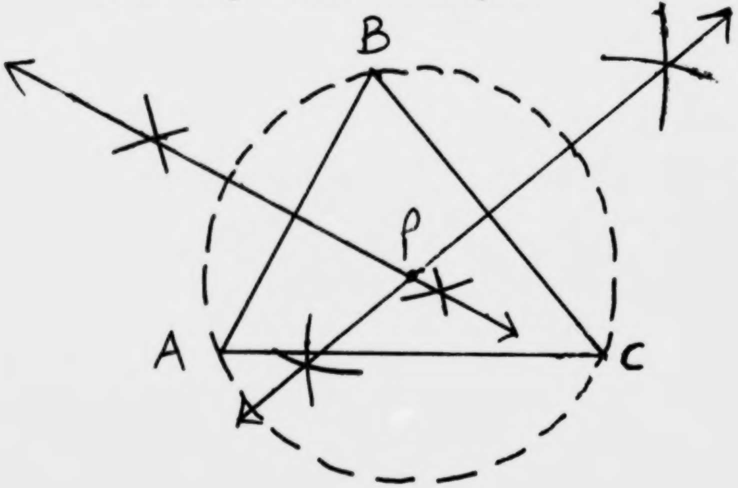
1. Construct a circle with radius 1 inch. Draw a triangle inside the circle such that the sides of the triangle are chords of the circle.
2. Draw any acute triangle ABC. Is it possible to construct a circle about the triangle which passes through all three vertices?
   Elicit that the center of the circumscribed circle must be equidistant from each of the vertices.

DEVELOPMENT:

1. Using the motivation problem, formally define inscribed polygon and circumscribed circle. Direct students' attention to the problem of finding the center of the required circle. Elicit that the problem may be restated as follows: to find the locus of points equidistant from three given noncollinear points. Now lead students to simplify the problem posed above by:

    1. Finding the locus of points equidistant from vertices A and B
    2. Finding the locus of points equidistant from vertices B and C
    3. Finding the locus of points equidistant from vertices A,B, and C

    Direct students to complete the construction by drawing the circle with center P, which is the locus of points equidistant from the three vertices, and with radius PA=PB=PC. Ask: How will this construction differ when the triangle is an obtuse triangle? Circumscribe a circle about any obtuse triangle.



2. Ask: Given the triangle having vertices A(3,2), B(7,2) and C(7,8), how can we locate the coordinates of the center of the circumscribed circle?

To find this point, elicit from students the procedure:

(1) Plot the points A, B, and C on a pair of axes.
(2) Find the locus of points equidistant from the two points A(3,2) and B(7,2).
(3) Find the locus of points equidistant from the two points B(7,2) and C(7,8).
(4) Find the locus of points equidistant from the three points A,B, and C, that is, the point (5,5).

Elicit that since the three given points lie in a coordinate plane, we need not do constructions to find the required loci. In addition, it is possible to find the equation of all loci using previously learned concepts.

Ask: 1) What is the equation of the locus of points equidistant from points A and B?
2) What is the equation of the locus of points equidistant from B and C?
3) What are the coordinates of the point at which these loci intersect?
4) What is the equation of the circle which can be circumscribed about triangle ABC?

3. Ask: Find the equation of the locus of points equidistant from points A(3,2) and B(7,10).

Ask: How is this problem different from the previous exercise?

Elicit that since $\overline{AB}$ is not parallel to one of the axes it is more difficult to find the equation of the locus. To resolve the problem, begin by outlining the following method.

a. Draw the segment AB and locate its midpoint M.
b. Find the slope of $\overline{AB}$ and its negative reciprocal.
c. Write the equation of the line perpendicular to $\overline{AB}$ and which contains point M.

Now have students complete the problem originally posed by solving graphically the system of equations representing the locus of points equidistant from A and B, and B and C.

APPLICATIONS:

1. Using graph paper find the equation of the locus of points equidistant from the following pairs of points
   (a) A(-3,5) and B(7,5)
   (b) A(5,-5) and B(5,-1)
   (c) A(-3,-6) and B(-3,6)
2. Find the equation of the locus of points equidistant from the points
   (a) A(2,5) and B(8,7)
   (b) A(8,0) and B(0,8)
3. Find the coordinates of the center of the circle that can be circumscribed about a triangle whose vertices are the points:
   (a) (-9,2), (-1,8), (-1,2)
   (b) (2,1), (6,11), (2,5)

SUMMARY:

1. State the steps which must be followed in constructing a circle circumscribed about a given triangle.
2. Is the center of the circle circumscribed about a given triangle always inside the triangle? Explain your answer.
3. The coordinates of the vertices of triangle ABC and A(3,4), B(13,4) and C(8,9).
   (a) Find the equation of the locus of points equidistant from points A and B.
   (b) Is C equidistant from A and B? Explain.
   (c) Find the distance from C to $\overline{AB}$.
   (d) Find the locus of points equidistant from points A, B and C.
   (e) Find the equation of the circle circumscribed about triangle ABC.
   (f) What kind of triangle is triangle ABC?

HOMEWORK:

1. Draw the given points on graph paper and find the equation of the locus of points equidistant from the given pairs of points.
   (a) (2,0) and (6,0)
   (b) (-7,2) and (-1,2)
   (c) (4,-2) and (4,8)
   (d) (2,1) and (4,5)
   (e) (3,3) and (7,-7)

2. Construct the circle circumscribed about a given right triangle.

3. The coordinates of the vertices of triangle ABC are A(2,1), B(8,1) and C(6,5).

   (a) Find the coordinates of M the midpoint of $\overline{AB}$
   (b) Write the equation of the locus of points equidistant from A and B.
   (c) Find the coordinates of N the midpoint of $\overline{BC}$.
   (d) Write the equation of the locus of points equidistant from B and C.
   (e) Find the coordinates of O the midpoint of $\overline{AC}$.
   (f) Write the equation of the locus of points equidistant from A and C.
   (g) Find the coordinates of the center of the circle which can be circumscribed about triangle ABC and write its equation.
4. Give the coordinates of a point on the y axis that is equidistant from the points P(-3,-3) and Q(-3,7).

5. The equation $x=5$ represents the locus of points equidistant from which of the following pairs of points?
   (a) (0,5) and (5,0)
   (b) (1,6) and (1,4)
   (c) (3,2) and (7,2)
   (d) (3,0) and (8,0)

LESSON 26

AIM: How do we find the equations of the locus of points equidistant from a given line, equidistant from two given parallel lines, and equidistant from two given intersecting lines?

PERFORMANCE OBJECTIVES: The students will be able to ....

1. Determine the locus of points equidistant from a given line to be a pair of parallel lines, draw the locus, and write the equations for the parallel lines, for cases where these lines are parallel to the coordinate axes.
2. Determine the locus of points equidistant from two given parallel lines to be a third line, parallel to the other two lines and midway between them, draw the locus, and write the equation of this third parallel line.
3. Determine the locus of points equidistant from two given intersecting lines to be the bisectors of the angles formed by these lines, draw the locus, and write the equations for the angle bisectors, for cases where these bisectors are parallel to the coordinate axes.

VOCABULARY: Equidistance between two parallel lines

MOTIVATION:

Ask: How many points can you find which are 3 units from the line whose equation is $x=4$? Give the coordinates of all points found.

DEVELOPMENT:

1. Elicit during a discussion of the motivation problem that there are an infinite number of points which are 3 units from the given line since the locus of points at a given distance from a given line is a pair of lines parallel to the given line and at the given distance from it. After sketching the locus, have students find the equations of the set of points 3 units from the line having the equation x=4.

   Ask students to find the equation of the locus of points
   (a) 2 units from the line whose equation is y=5
   (b) 5 units from the line whose equation is x=-2
   (c) 6 units from the x axis

2. Direct students to draw on the same set of axes, the graphs of y=2x+1 and y=2x-5. Ask: How many points can you find which are equidistant from the two given lines? Give the coordinates of all such points.

   Elicit from students that there are an infinite number of points equidistant from the two given parallel lines since the locus of points equidistant from two parallel lines is a third line parallel to the two given lines and midway between them. In this case the line will have a slope of 2 and a y intercept midway between the points (0,1) and (0,-5), the points where the given lines intersect the y axis.

3. Ask students to draw the graph and find the equation of the locus of points equidistant from the two lines with equations:
   - (a) $x=2$ and $x=10$
   - (b) $x=-3$ and $x=3$
   - (c) $y=-2$ and $y=6$
   - (d) $y=1/2\ x + 8$ and $y=1/2\ x - 2$

4. Ask: Can we use the method just learned to find the locus of points equidistant from the lines whose equations are $y=2x$ and $y=-2x$?

   Elicit since the lines are intersecting, not parallel, we must use the locus principle which states the locus of points equidistant from two given intersecting lines is the pair of lines which bisect the angles formed by the given line.

   Direct students to study the graphs of the given lines to determine the equation of the desired locus. ($y=0$ and $x=0$)

APPLICATIONS:

Write the equation of the locus of points
- (a) 5 units from the y axis
- (b) 4 units from the line with equation $y=1$
- (c) equidistant from the lines whose equations are $x=-5$ and $x=7$
- (d) equidistant from the lines having equations $y=4x-5$ and $y=4x+5$
- (e) equidistant from the lines having equations $y=-3/2\ x$ and $y=3/2\ x - 6$

SUMMARY:

1. State a procedure which can be used to find the equation of the locus of points equidistant from two given parallel lines if
   - (a) the lines are parallel to an axis
   - (b) the lines are not parallel to an axis

2. Write the equation of the locus of points which are the centers of circles of radius 2 which are tangent to the x axis.

3. Write the equation of the locus of points which are the centers of circles which are tangent to both the x and y axes.

HOMEWORK:

1. Graph and find the equation of the locus of points
   - (a) four units from the line with equation $x=3$.
   - (b) equidistant from the lines with equations $y=5$ and $y=-3$
   - (c) one and one-half units from the line with equation $y=3/2$
   - (d) equidistant from the lines with equations $x=-6$ and $x=-2$
   - (e) equidistant from the lines with equations $y=x$ and $y=x-6$
   - (f) 3 units from the x axis
   - (g) equidistant from the x and y axes

2. (a) On graph paper draw the lines whose equations are $y = 1/3\ x + 2$ and $y = -1/3\ x + 8$.
   (b) Draw the locus of points equidistant from these intersecting lines.
   (c) Write the equation of the line drawn in part b.

3. The vertices of triangle ABC and A(4,3), B(12,3), and C(8,8). Write the equation of the locus of points which are the vertices of triangles equal in area to triangle ABC and which have AB as their base.

4. Write the equation of the locus of points equidistant from the circles with equations $x^2 + y^2 = 100$ and $x^2 + y^2 = 16$.

LESSON 27

AIM: How can we describe the locus of points equidistant from a given point and a given line?

PERFORMANCE OBJECTIVES: The students will be able to ....

1. Determine that the locus of points equidistant from a given point and a given line is a smooth curve, called a parabola.
2. State the definition of parabola, focus, and directrix.
3. Find the equation of a parabola, using the distance formula given the coordinates of the focus and the equation of the directrix, where the focus is any point in the plane and the directrix is a line parallel to one of the coordinate axes.
4. Write the equation of a parabola in the form $y=ax^2+bx+c$ or the form $x=ay^2+by+c$, according to the position of its directrix, where a,b and c are real numbers and $a \neq 0$.
5. Graph a parabola, given the coordinates of its focus and the equation of its directrix (parallel to one of the coordinate axes)

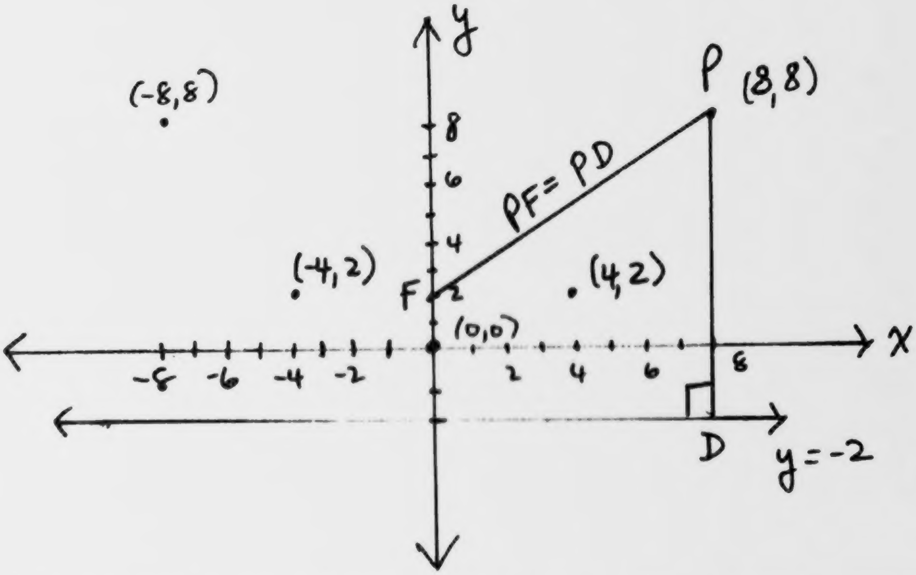
VOCABULARY: parabola, focus, vertex, directrix

MOTIVATION:

Given the point F(0,2) and the line whose equation is $y=-2$ find the coordinates of five points equidistant from the point F and the given line.

DEVELOPMENT:

1. Review with the class the distance from a point to a line is the length of the perpendicular segment from the point to the line. After sketching the given point and the given line on a pair of coordinate axes, use colored chalk to identify points which are equidistant from the point F and the line with equation y=-2.
   (Easily understood points are (0,0) (4,2) and (-4,2))



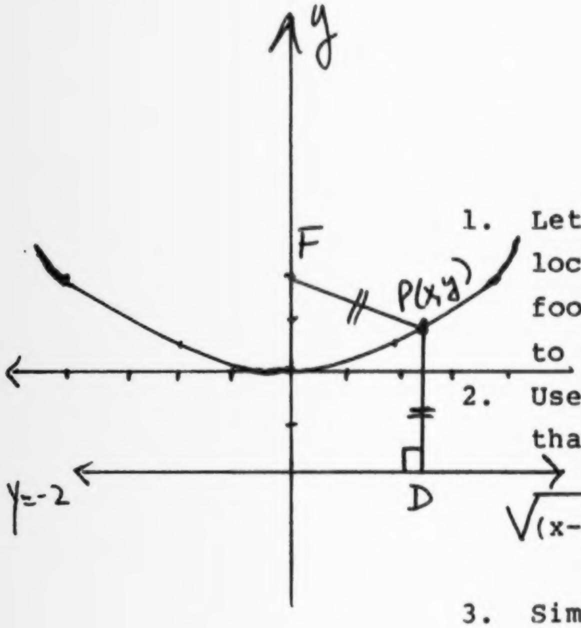
Ask students to select more points which appear to be equidistant from the given point and the given line and to imagine what the locus would look like.

Tell students the graph they have described is called a parabola, which is defined to be the locus of points equidistant from a given line and a given point not on the line. The fixed point is called the focus and the given line is called the directrix.

Explain that parabolas are important curves which are found in the arching shape of many bridges, as well as in the paths of moving objects, such as rockets, comets and baseballs. Even cross sections of radar reflectors and searchlights are shaped like parabolas because of the special reflective properties of these curves.

2. Using the motivation problem, ask students: How can we find the equation of this parabola?

Suggest students use the definition just learned to derive the equation of the parabola with focus F(0,2) and directrix with equation y=-2.



1. Let P (x,y) be any point on the locus, then PD=PF, where D is the foot of the perpendicular from P to the line with equation y=-2.
2. Use the distance formula to show that

$$\sqrt{(x-0)^2 + (y-2)^2} = \sqrt{(y+2)^2}$$

3. Simplify by squaring both sides:

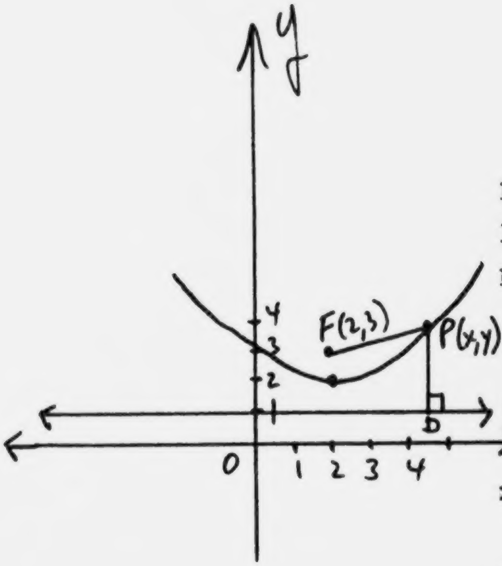
$$x^2 + (y-2)^2 = (y+2)^2$$

4. Multiply and combine to obtain $y = 1/8\ x^2$, the equation of the parabola with focus F(0,2) and directrix y=-2.

Have students find other points of the graph by finding coordinates which satisfy the equation just found. They can then draw an accurate graph of this parabola.

3. Ask: What is the equation of the locus of points equidistant from the point F(2,3) and the line whose equation is y=1.

Elicit from students that the same procedure must be followed as in part (1) to find the equation of this locus.



Let P(x,y) be any point on the locus. then

PF=PD

$(x-2)^2+(y-3)^2= (y-1)^2$

$(x-2)^2+(y-3)^2=(y-1)^2$

$x^2-4x+12 = 4y$

$y=1/4\ x^2 -x +3$

is the equation of the locus of points equidistant from the point (2,3) and the line whose equation is y=1.

APPLICATIONS:

1. Given the focus and directrix in each exercise below
   (a) Find the coordinates of three points which are equidistant from the focus and directrix.
   (b) Using the three points found in part (a), sketch the parabola.

      (1) Focus (2,3) , directrix y=4
      (2) Focus (3,2) , directrix x=-1
      (3) Focus (2,1) , directrix x=6
      (4) Focus (-3,0) , directrix y=-2

2. Find the equation of each parabola sketched in number 1.

Summary;

1. Given a parabola with focus (q,p) and equation of directrix y=k, sketch the parabola and write its equation in general form if (a) k>p (b) k<p.

2. Given a parabola with focus (p,r) and equation of directrix x=k, sketch the parabola and write its equation in general form if (a) p>k (b) p<k.

HOMEWORK:

1. Sketch part of the parabola which has the given focus F and the directrix with the given equation by first finding three points equidistant from the given focus and the given directrix.

   (a) Focus (1,6) and directrix y=0
   (b) Focus (0,0) and directrix y=-6
   (c) Focus (-2,-7) and directrix y=-1
   (d) Focus (-2,-3) and directrix x=2
   (e) Focus (5,0) and directrix x=-3

2. Find an equation, in the form $y=ax^2+bx+c$, of the parabola with focus and directrix having the given equation:
   (a) F(0,-7) , y=-3
   (b) F(4,2) , y=0

3. Find an equation in the form $x=ay^2+by+c$ of the parabola with focus F and directrix having the given equation:
   (a) F(5,1) , x=1
   (b) F(-7,3) , x=-3

LESSON 28

AIM: How do we find the points in the coordinate plane which satisfy two different sets of conditions?

PERFORMANCE OBJECTIVES: The students will be able to ....

1. Determine the intersection of loci in the coordinate plane by sketching on graph paper the given conditions.

VOCABULARY: No new terms

MOTIVATION:

Describe fully the locus of points which are equidistant from the point A(5,4) and the x-axis and which are also 4 units from the point A(5,4).

DEVELOPMENT:

1. A discussion of the motivation problem will elicit the need for a carefully drawn diagram on graph paper when locus problems involve coordinates.

   Elicit the same procedure used previously in finding the points in a plane which satisfy two different sets of conditions must be used in this problem.

   (a) Find the locus of points equidistant from the point A(5,4) and the x-axis to be the parabola with focus A and directrix having equation y=0.

(b) Find the locus of points 4 units from the point A(5,4) to be the circle with center A and radius 4.

(c) Find the set of points which are the intersection of the two loci drawn in parts (a) and (b), which are the two points P(1,4) and Q(9,4).

Ask: Why is the solution to the motivation problem the intersection of the loci found in answer to parts (a) and (b)?

2. Direct students to find loci involving variable distances, for example:

Given the point P whose coordinates are (2,5)

(a) Describe fully the locus of points d units from P.

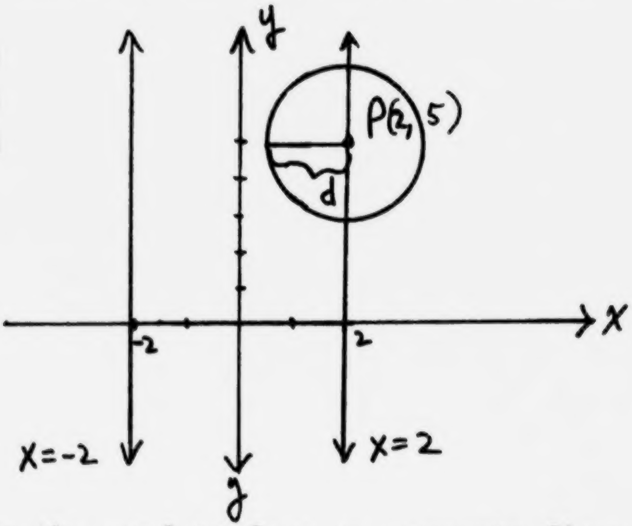
(b) Describe fully the locus of points 2 units from the x-axis.

(c) How many points satisfy the conditions in parts (a) and (b) simultaneously for the following values of d?

(1) d = 3

(2) d = 5

(3) d = 8



Elicit from students the need to draw an accurate diagram on graph paper for each case. Use color chalk to sketch the loci.

3. Show how graph paper can be used to help in finding loci involving numbers, not necessarily coordinates. For example, ask: How many points are there in a plane that are 3 units from a given line and 5 units from a given point on that line?

Elicit that this problem can be done without graph paper, but using the lined graph paper will make our work easier and more accurate.

APPLICATION:

1. Find the locus of points equidistant from the lines with equations $x=-2$ and $x=4$ and also equidistant from the lines having equations $y=2$ and $y=6$.

2. $\overline{AB}$ and $\overline{CD}$ are two parallel lines 8 units apart, and R is a point in $\overline{AB}$. How many points are there that are equidistant from $\overline{AB}$ and $\overline{CD}$ and 3 units from R?

3. (a) Draw the locus of points equidistant from the points $A(-2,3)$ and $B(6,3)$ and write the equation of this locus.
   (b) Draw the locus of points 4 units from the line with equation $y=1$ and write the equation(s) of this locus.
   (c) Find the number of points that satisfy both the conditions stated in parts (a) and (b). Give the coordinates of each point found.

SUMMARY:

1. Outline a procedure which can be used to find the locus of points which satisfy two different sets of conditions.

2. The vertices of triangle ABC are $A(3,2)$, $B(11,2)$ and $C(11,10)$.

   (a) Find the locus of points equidistant from points A and B and write its equation.
   (b) Find the locus of points equidistant from points B and C and write its equation.

(c) Find the locus of points equidistant from points A, B and C and write its coordinates.

(d) Write the equation of the circle which circumscribes triangle ABC.

HOMEWORK:

1. The number of points equidistant from both the x axis and the y axis and also 5 units from the origin is

   (a) 1 (b) 2 (c) 3 (d) 4

2. Parallel lines m and p are 8 centimeters apart and point A is on line m. The number of points equidistant from m and p and also 5 units from point A is

   (a) 0 (b) 1 (c) 2 (d) 3

3. How many points are 2 units from line segment RS and 1 unit from R?

   (a) 0 (b) 1 (c) 2 (d) 3

4. (a) Represent graphically the locus of points 4 units from the line whose equation is $y=3$ and 2 units from the line whose equation is $x=4$.

   (b) Write the equation of the loci represented in (a).

   (c) Find the coordinates of the points of intersection of the loci represented in answer to (a).

5. The vertices of triangle ABC are A(-2,1), B(-2,11) and C(8,5).

   (a) Draw the graph and write the equation of the locus of points equidistant from A and B.

   (b) Draw a line CD that is parallel to $\overline{AB}$ and write an equation of $\overline{CD}$.

   (c) Write the coordinates of the points of intersection of the lines drawn in parts (a) and (b).

6. (a) Draw the locus of points equidistant from the circles whose equations are $x^2+y^2=9$ and $x^2+y^2=25$. Write an equation of the locus.

(b) Draw the locus of points 4 units from the y axis. Write equation(s) of the locus.

(c) Find the number of points that satisfy both conditions stated in (a) and (b). Write the coordinates of each point found.

7. (a) Using graph paper, draw the triangle whose vertices are A(3,3), B(9,3) and C(7,7).

(b) Draw the locus of points equidistant from A and B.

(c) Write an equation of the line drawn in answer to (b).

(d) Write the equation of the line passing through points A and C.

(e) Find the coordinates of point P if P is a point in the line AC that is equidistant from points A and B.

8. (a) Given points P(-2,3) and Q(4,3), draw the graph of the locus of points equidistant from P and Q.

(b) Write the equation of the locus of points drawn in answer to (a).

(c) On the same set of axes used in part (a), draw the graph of the locus of points 5 units from Q.

(d) Write the equation of the locus drawn in answer to (c).

(e) What is the total number of points that satisfy the conditions described in parts (a) and (c) simultaneously?

(f) Find the coordinates of all points found in answer to part (e).

Lesson: 29

AIM: How do we solve quadratic equations by factoring?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define quadratic equation.
2. Transform a quadratic equation into standard form.
3. Factor the greatest common monomial factor of a polynomial expression.
4. Factor the trinomial $ax^2 + bx + c$ into two binomial expressions.

VOCABULARY: quadratic equation, quadratic polynomial, greatest common monomial factor, trinomial

MOTIVATION:

The area of a rectangle is $2X+6$. What are its possible dimensions?

DEVELOPMENT:

Elicit that one set of dimensions is length = $x+3$ and width = 2 (there are others).

Since the product, $2x + 6$, can be obtained by multiplying 2 and $x + 3$, then 2 and $x + 3$ are the factors of $2x + 6$. If $2x + 6$ were given as the product of two factors, how would we determine those factors? Have students see that the binomial, $2x + 6$, consists of two terms, $2x$ and 6. Factors of $2x$ are 2 and x. Factors of 6 are 2 and 3. Because 2 is a factor of $2x$ and 6, it is called the common factor of $2x$ and 6. Then, $2x + 6 = 2 \cdot x + 2 \cdot 3$. Using the distributive property, we can write this as $2x + 6 = 2(x + 3)$.

This sentence is true for any replacement of the variable by a signed number. The given polynomial has 2 and $x + 3$ as factors.

In a similar way, have students show that $8x^2 - 12x = 4x(2x - 3)$.

Emphasize that $4x$ is the greatest common monomial factor since it has the greatest numerical coefficient and the greatest degree common to each term. In expressing a polynomial as the product of factors, one of which is a monomial factor, it is understood that we use the greatest common monomial factor.

Have students complete the following table:

POLYNOMIAL	GCF	POLYNOMIAL ÷ GCF = QUOTIENT	FACTORED FORM
$5x^2 - 10$	5	$\frac{5x^2}{5} - \frac{10}{5} = x^2 - 2$	$5(x^2 - 2)$
$9a^2x - 18ax^2$			
$12a^3 b + 16a^2 b^2$			
$\pi r^2 + \pi rl$			
$ab + b$			

A polynomial that has the form ax^2+bx+c $(a \neq 0)$ is called a quadratic polynomial since the highest power of the variable is 2. X^2+5x+8, $3x^2-7$, and $-2x^2+13x$ are examples of quadratic polynomials.

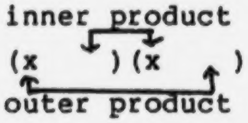
An equation such as $X^2+6x+8=0$ is called a quadratic equation. When a quadratic equation is written so that one side is zero and the other is a quadratic polynomial

in which the terms are arranged in descending power (exponent) order, the quadratic equation is in standard form.

Standard Form: $ax^2+bx+c=0$, where a, b, and c are real numbers and $a \neq 0$.

One method of solving quadratic equations is based on factoring the quadratic polynomial. Today we will learn to factor quadratic polynomials. Consider X^{2+7x+6}.

We know that the factors are two binomials.
The product of the first two terms in each binomial must be equal to the first term in the trinomial so we can write $x^2+7x+6=(x\quad)(x\quad)$.
The product of the last terms in each binomial must be equal to the last term in the trinomial. In this example, we have a choice of +1 and +6, -1 and -6, +2 and +3, -2 and -3.
The algebraic sum of the inner product and the outer product of the two binomials



must equal the middle term of the trinomial. Using the pairs of factors given above, only +1 and +6 give an inner product and outer product whose sum is +7x. The correct factors are (x + 1)(x + 6).

(1) Factor trinomials of the type $x^2 + bx + c$ where $c>0$ and $b\neq 0$. Elicit from the class that if $c>0$, then the last term in each factor will both be positive or they will both be negative. If $b>0$ they will both be positive. If $b<0$ they will both be negative.
(2) Factor trinomials of the type $x^2 + bx + c$ where $c<0$, $b\neq 0$. Elicit from the class that if $c<0$, then the signs of the second term in each binomial factor must be different.

(3) Factor completely $4x^2 - 8x - 12$. GCF is 4.
$4(x^2 - 2x - 3)$
$4(x - 3)(x + 1)$

To check: multiply the factors.

APPLICATIONS:

Factor: $x^2 + 8x + 15 = (x + 3)(x + 5)$
$x^2 - 12x + 27 = (x - 9)(x - 3)$
$x^2 - 6x + 8 = (x - 4)(x - 2)$
$b^2 - 13b - 48 = (b + 3)(b - 16)$
$r^2 - r - 72 = (r - 9)(r + 8)$
$3x^2 - 2x = x(3x - 2)$
$3x^2 - 6x = 3x(x - 2)$
$3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1)$
$x^2 + x - 20 = (x + 5)(x - 4)$

SUMMARY:

If $x^2 + bx + c$ is factorable into $(x + r)(x + s)$, what is true of the signs of r and s, if c is a positive number? What is true of the signs of r and s if c is a negative number? If c is a positive number, how does the sign of b affect the signs of r and s? If c is negative number, how is the sign of b related to the absolute values of r and s?

HOMEWORK:

1. Factor: $4m^2 - 20$
$3a^3 - 5a^2$
$4m^3n^2 - 32m^2n$
$5a^3b^2 - 35ab$
$8a^2 - 16a + 40$
$m^3 - m^2 - m$

2. Factor: $x^2 + x - 6$
$x^2 - 7x + 12$
$a^2 + 4a + 3$
$a^2 - 7a - 18$
$y^2 - 2y - 15$
$x^2 + 2x + 1$

3. Factor completely:

$3x^2 + 3x - 18$
$4y^2 - 8y - 60$
$5x^2 + 10x + 5$
$3x^2 - 6x - 9$

4. *Factor: $x^2 + 8x + 15$
$3x^2 - 6x - 9$
$x^2 - 4x$
$x^2 - x - 12$
$x^2 + 3x - 10$

*Examples will be used in the next lesson.

Lesson: 30

AIM: How do we solve quadratic equations by factoring?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Factor a quadratic equation and set each factor having a variable equal to zero.
2. Solve each resulting equation and check the values obtained in the original equation.

VOCABULARY: no new terms

MOTIVATION:

What value(s) of x will make each of the following equations true?

(a) $3x = 0$
(b) $5(x - 2) = 0$
(c) $(x - 1)(x + 2) = 0$

What do these products have in common? They are equal to 0 so that at least one of the factors must be 0. This property may be stated formally as follows:

For all real numbers a and b,
if $ab = 0$, then $a = 0$ or $b = 0$ or a and $b = 0$

DEVELOPMENT:

Consider yesterday's homework assignment when you were asked to factor $x^2 + 8x + 15$. Using that information solve:

$$x^2 + 8x + 15 = 0$$

$$(x + 3)(x + 5) = 0$$

$$\begin{array}{l|l}
x + 3 = 0 & x + 5 = 0 \\
\underline{\quad -3 \quad -3} & \underline{\quad -5 \quad -5} \\
x \qquad = -3 & x \qquad = -5
\end{array}$$

Check each value in the original equation.

$x^2 + 8x + 15 = 0$	$x^2 + 8x + 15 = 0$
$(-3)^2 + 8(-3) + 15 \overset{?}{=} 0$	$(-5)^2 + 8(-5) + 15 \overset{?}{=} 0$
$9 - 24 + 15 \overset{?}{=} 0$	$25 - 40 - 15 \overset{?}{=} 0$
$24 - 24 \overset{?}{=} 0$	$40 - 40 \overset{?}{=} 0$
$0 = 0$ True	$0 = 0$ True

1. Solve: $x^2 + 3x = 10$

   Since the basic principle underlying the solution of a quadratic equation by factoring involves a product which is equal to zero, have students realize that the equation must be written so that one side is zero.

   $x^2 + 3x = 10$

   $x^2 + 3x - 10 = 0$ Why is this equivalent?

   Have students complete the solution. Have them check.

   Answers: $\{-5, 2\}$

2. Solve: $x^2 - 4x = 0$

   $x(x - 4) = 0$

   Then $x = 0$ or $x - 4 = 0$ and $x = 4$, or both. Why?

   Then the solution set is $\{0, 4\}$.

   Check each apparent solution in the original equation.

$0^2 - 4(0) \overset{?}{=} 0$	$(4)^2 - 4(4) \overset{?}{=} 0$
$0 - 0 \overset{?}{=} 0$	$16 - 16 \overset{?}{=} 0$
$0 = 0$	$0 = 0$

   Why is $x^2 = 4x$, $x = 4$ not a correct method? (We cannot divide both sides of an equation by a variable if zero is a possible replacement for that variable.)

3. Solve: $x^2 - 12 = x$

   $x^2 - x - 12 = 0$

   $(x - 4)(x + 3) = 0$

   Then $x - 4 = 0$ or $x + 3 = 0$ and

   $x = 4$ and $x = -3$, or both.

   Then the solution set is $\{4, -3\}$.

   Check each apparent solution in the original equation, $x^2 - x - 12 = 0$.

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4. $$3x^2 - 9 = 6x$$
$$3x^2 - 6x - 9 = 0$$
$$3(x - 3)(x + 1) = 0$$
$$\begin{array}{c|c} x - 3 = 0 & x + 1 = 0 \\ \underline{\quad +3 \quad +3} & \underline{\quad -1 \quad -1} \\ x = 3 & x = -1 \end{array}$$

Check each value in original equation.

Discuss with students the number of solutions they expect a first-degree equation to have; a second degree equation to have; a third degree equation to have.

In order to transform a quadradic equation into standard form, it may be necessary to perform one of the following steps.

1. CLEAR OF FRACTIONS: The equation $3x = \frac{8}{x} + 4$ $(x \neq 0)$ is cleared of fractions by multiplying each side by x.

   $3x^2 = 8 + 4x$ $\qquad$ $3x^2 - 4x - 8 = 0$

2. REMOVE PARENTHESES: Consider $2(x^2 - 7) = 12x$. Apply the distributive law.

   $2x^2 - 14 = 12x$ $\qquad$ $2x^2 - 12x - 14 = 0$

3. REMOVE RADICAL SIGNS: Consider $x + 1 = \sqrt{3x + 7}$; square each side.

   $$(x + 1)^2 = 3x + 7$$
   $$x^2 + 2x + 1 = 3x + 7$$
   $$x^2 - x - 6 = 0$$

4. COLLECT LIKE TERMS: Consider $x^2 + 2x + 1 = 3x + 3$; collect and combine like terms.

   $$x^2 + 2x + 1 - 3x - 3 = 0$$
   $$x^2 - x - 2 = 0$$

SUMMARY:

Solve $(x - 4)(x - 3) = 2$ by writing $x - 4 = 2$ and $x - 3 = 2$. Ask students to comment. Have students realize that although the polynomial on the left side of the equation is already in factored form, the product of the factors is not zero, but 2. Therefore, the zero product principle cannot be applied. To solve the equation, we must find an equivalent equation written so that one or both of the factors is zero.

Have students proceed as follows:

$(x - 4)(x - 3) = 2$

$x^2 - 7x + 12 = 2$

$x^2 - 7x + 10 = 0$

$(x - 5)(x - 2) = 0$

Then $x - 5 = 0$ | or $x - 2 = 0$

and $x = 5$ | and $x = 2$

Then the solution set is $\{5,2\}$.

Check: $(5 - 4)(5 - 3) \overset{?}{=} 2$ | $(2 - 4)(2 - 3) \overset{?}{=} 2$

$1\cdot 2 \overset{?}{=} 2$ | $(-2)(-1) \overset{?}{=} 2$

$2 = 2$ | $2 = 2$

APPLICATIONS:

Solve and check

1. $x^2 - 8x + 12 = 0$
2. $x^2 - 7x = 0$
3. $x^2 + 4x - 5 = 0$
4. $5y^2 + 10y + 5 = 0$
5. $a^2 - 10 = 3a$
6. $7x = -x^2 - 10$
7. $-x^2 = 3x$
8. $2x - 1 - \frac{14}{x} = 11$
9. $3x(x-2)-2 = 7$

HOMEWORK:

Solve and check:

1. $p^2 + 5p - 6 = 0$
2. $m^2 + 4m - 12 = 0$
3. $3x^2 + 3x - 18 = 0$

Find the solution set for each of the following:

4. $x^2 - 9x + 20 = 0$
5. $a^2 + 4a + 3 = 0$
6. $a^2 - 7a - 18 = 0$

Solve and check:

7. $-x^2 = 9x$
8. $18 - 7b = b^2$
9. $8x = -x^2 - 15$

10 $\frac{1}{2x} - \frac{1}{3} = \frac{x}{6}$

Lesson: 31

AIM: How do we solve quadratic equations which are incomplete, are perfect square trinomials or have leading coefficients other than 1? (PartI)

PERFORMANCE OBJECTIVES: The student will be able to...

1. Identify quadratic equation as incomplete.
2. Factor the difference of two squares.
3. Determine when an incomplete quadratic equation does not have real roots.
4. Define a perfect square trinomial.
5. Factor a perfect square trinomial.

VOCABULARY: perfect square trinomial, difference of two squares

MOTIVATION:

Have students compare the following equations.

1) $x^2 + 3x + 2 = 0$ 2) $x^2 + 2 = 0$

How are they alike? (Each is a second degree or quadratic equation. The second degree terms are the same, as are the constant terms.)

How are they different? (There is no first degree term in equation 2.)

Tell students that a quadratic equation is said to be incomplete if the linear or first degree term is missing ($b = 0$). $ax^2 + c = 0$ where a and c are real numbers, $a \neq 0$, is an incomplete quadratic equation.

Have students identify the incomplete quadratic equations among the following.

1) $x^2 - 49 = 0$ 3) $a^2 = a + 6$

2) $x^2 + 9x + 8 = 0$ 4) $2y^2 = 50$

DEVELOPMENT:

1. Have students find the following products.

a) $(x+5)(x-5) = (x+5)x + (x + 5)(-5)$ Distributive property

$= x^2 + 5x + (-5x) + (-25)$ Distributive

$= x^2 - 25$ The sum of the number and its additive inverse is zero.

b) $(x+2)(x-2) = x^2 - 4$

c) $(-x+7)(-x-7) = x^2 - 49$

What is true about the binomial factors in these examples? The first terms in each pair of binomials are the same and the second terms are additive inverses. Why is there no first degree term containing x in the products?

Have students realize that since the product of two binomials whose first terms are the same and whose second terms are additive inverses results in a binomial which is the difference of two squares. We may reverse the process to factor the difference of two squares. We factor the difference of two squares of monomials into the sum and the difference of the monomials.

2. Can $x^2 - 9$ be factored? Here we have a binomial which is a difference of two squares, neither of which is equal to
$x^2 - 9 = (x)^2 - (3)^2 = (x-3)\ (x+3)$
Multiplication of the binomial factors will verify this statement. A more general result may be obtained. The factors of any binomial of the form $a^2 - b^2$ are $(a + b)$ and $(a - b)$.

3. The binomial factors for a binomial which is the difference of two squares appear very much like the factors of trinomials which were studied in the previous lessons. Why do we not obtain a trinomial when the factors are multiplied?

inner product
$(a + b)(a - b)$
outer product

The inner product and outer product are additive inverses. The sum of $ab + (-ab)$ is equal to zero. Therefore, the product is the binomial, $a^2 - b^2$.

4. Solve: $z^2 = 16$. Have students realize that if $z^2 = 16$, then $z = \pm\sqrt{16} = +4$ or -4. These are the solutions. Will the solution by factoring give the same answers?

$$z^2 - 16 = 0$$

$$(z + 4)(z - 4) = 0$$

$$\begin{array}{l|l} \text{Then } z + 4 = 0 & \text{or } z - 4 = 0 \\ \text{and } z = -4 & \text{and } z = 4 \end{array}$$

The solution set is $\{-4, 4\}$. Have students check.

5. Solve for all real roots: $x^2 + 4 = 0$.
   a) $x^2 + 4$ cannot be factored for real numbers.
   b) $x^2 + 4 = 0$, $x^2 = -4$, there is no real number which when squared $= -4$. $x^2 + 4 = 0$ has no real roots.

Indicated that in general, if $x^2 = c$, then $x = \pm\sqrt{c}$; if $c < 0$, then x is not a real number.

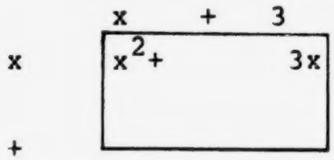
6. Solve for all real roots:

(a) $x^2 - 13 = 0$	(c) $5x = 15x^2$	e) $2-3x^2 = 26$
(b) $x^2 - 7x = 0$	(d) $(x - 2)^2 = 36$	

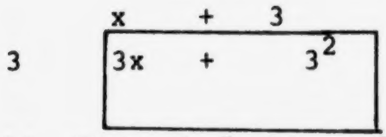
Have students consider the special case of multiplying a binomial by itself.

a)
$$\begin{aligned}(x + 3)^2 &= (x + 3)(x + 3) \\ &= (x + 3)x + (x + 3)3 \\ &= x^2 + 3x + 3x + 9 \\ &= x^2 + 6x + 9\end{aligned}$$

Have students note that the area of the square whose side is $(x + 3)$ is made up of the areas of 2 squares and 2 identical rectangles.



$$= x^2 + 2(3x) + 3^2$$



b) Have students perform several multiplications of identical binomials using the usual method of multiplication.

Factored Form	Product
$(y + 5)(y + 5)$ or $(y + 5)^2$	$= y^2 + 10y + 25$
$(a - 2)(a - 2)$ or $(a - 2)^2$	$= a^2 - 4a + 4$
$(2b + 1)(2b + 1)$ or $(2b + 1)^2$	$= 4b^2 + 4b + 1$

c) Pose questions:

a. What kind of polynomial is the product? (a trinomial)

b. How does the first term of the product compare with the first term of the binomial in each case? (It is the square of the first term of the binomial.)

c. How would you describe the last term of each product?

d. How would you describe the second term of each product? (It is twice the product of the terms of the binomial.)

Tell students that a trinomial which is the square of a binomial is called a perfect square trinomial or a trinomial square.

d) Have students factor trinomial squares.

a. Pose question: How can we recognize a trinomial that is the square of a binomial? For example, is $x^2 + 12x + 36$ a trinomial square?

Have students recall the relationship between the terms of the product which is the square of a binomial and the terms of the binomial. Is the first term of the trinomial a square? Yes, x^2 is the square. Is the last term a square? Yes, 36 is 6^2. Is the middle term twice the product of x and 6? Yes, $12x = 2(x)(6)$. Have students conclude that the trinomial is the square of a binomial.

b. Lead students to see that the trinomial $x^2 + 12x + 36$ may then be written in factored form as: $x^2 + 12x + 36 = (x + 6)(x + 6)$ or $(x + 6)^2$.

c. In similar fashion, have students see that $a^2 - 14a + 49$ is a trinomial square.

7. Solve: $x^2 - 8x + 16 = 0$.

$$(x - 4)(x - 4) = 0$$

Then $x - 4 = 0$ | or $x - 4 = 0$

and $x = 4$ | and $x = 4$

Have students note that since the factors are identical, we obtain the same root, 4, twice. It is therefore called a double root, but is written only once as a member of the solution set. The solution set is $\{4\}$. Have students check.

APPLICATIONS:

1. Factor:

   $x^2 - 36$

   $36 - y^2$

   $4m^2 - 25$

   $81y^2 - 16$

2. Tell which are the differences of two squares.

   $x^2 - 16$ YES

   $169 - 3y^2$ NO

   $m^2 + 49$ NO

   $y^2 - 35$ NO

   $25a^2 - 121$ YES

3. Factor:

   $x^2 + 6x + 9$

   $x^2 + 10x + 25$

   $x^2 + 16x + 64$

4. Solve (a) $x^2 + 14x + 49 = 0$

   (b) $x^2 = 17$

   (c) $12x^2 = 6x$

   (d) $4x - 18x^2 = 16x$

SUMMARY:

1. How can we recognize a trinomial that is the square of a binomial?

2. Why do we not obtain a trinomial when $(a + b)(a - b)$ are multiplied?

3. The area of a square is $9x^2 - 6x + 1$. Express the length of a side as a binomial in terms of x.

HOMEWORK:

1. **Factor:**
   $a^2 - 81$
   $9a^2 - 25$
   $49y^2 - 4$
   $25 - x^2$
   $b^2 - 25$
   $b^2 - 1$
   $y^2 - 64$
   $49x^2 - 100$

2. **Factor:**
   $x^2 - 20x + 100$
   $x^2 - 10x + 25$
   $x^2 + 6x + 9$
   $x^2 - 4x + 4$
   $4x^2 + 4x + 1$
   $4x^2 - 12x + 9$

3. Solve:
   (a) $x^2 - 12x + 36 = 0$
   (b) $11 - x^2 = 3$
   (c) $\frac{4}{x - 3} = \frac{x + 3}{4}$
   (d) $x^2 + 18x + 81 = 0$
   (e) $x^2 + 5x = 2x^2$
   (f) $14 - (x + 1)^2 = 5$

Lesson: 32

AIM: How do we solve quadratic equations which are incomplete, are perfect square trinomials or have leading coefficients other than 1? (Part II)

PERFORMANCE OBJECTIVES: The student will be able to...

1. Factor a quadratic polynomial whose leading coefficient is not 1.
2. Solve verbal problems involving quadratic equations.

VOCABULARY: no new terms

MOTIVATION:

The area of a rectangle is $3x^2 + 7x + 2$. What binomial factors might represent the dimensions of the rectangle?

DEVELOPMENT:

1. Following the same procedure used when the coefficient of the second degree term is 1, the student should list all possible factors, e.g., $(3x+1)(x+2)$, $(3x+2)(x+1)$. Finding the sum of the inner and outer products, they obtain either $+7x$ or $+5x$, thereby achieving the factors of $3x^2 + 7x + 2$ as $(3x + 1)(x + 2)$. This "trial and error" procedure is always used when factoring a trinomial where $a > 1$.
2. For the trinomial $6x^2 + 5x - 4$, we search for all possible binomial pairs which can be factors:
    a) (6x1)(x4)
    b) (6x4)(x1)
    c) (6x2)(x2)
    d) (3x1)(2x4)
    e) (3x4)(2x1)
    f) (3x2)(2x2)

   Have students see that since there is a common factor in the original expression, there can be no common factor in any of its factors. This eliminates choices b, c, d and f. Since the last term is -4, we have the factors

$(6x-1)(x+4)$ $(3x-4)(2x+1)$

$(6x+1)(x-4)$ $(3x+4)(2x-1)$

Multiplying each, we obtain the correct pair of factors: $(3x+4)(2x-1)$.

3. In factoring trinomials where $a > 1$ (such as those given below), point out that the sign of the last term in each binomial factor cannot be determined until the sum of the inner and outer products of the binomial factors is compared to the middle term of the trinomial.

4. Solve: $2x^2 - 8x + 6 = 0$. Have students see that there is a common monomial factor, 2. Since this is a non-zero number, we may divide each member of the equation by 2. Then we have

$x^2 - 4x + 3 = 0$	or $2(x - 4x + 3) = 0$
$(x-3)(x-1) = 0$	$2(x-3)(x-1) = 0$
Then $x - 3 = 0$ or $x - 1 = 0$	Since $2 \neq 0$ then $x - 3 = 0$
and $x = 3$ and $x = 1$	or $x-1 = 0$ and $x = 3$
The solution set is $\{3,1\}$	and $x = 1$

Have students check the solutions in the original equation.

5. Solve: $2x^2 + 9x - 5 = 0$. Have students determine the binomial factors by asking:
   a) What are the first terms of the binomial factors?
   b) What is the product of the last terms? What are the pairs of factors of -5?
   c) Which pair of factors would you select? Why?

   $(2x + 1)(x - 5)$ $(2x + 5)(x - 1)$

   $(2x - 1)(x + 5)$ $(2x - 5)(x + 1)$

   Ask the class to complete the solution of the equation.

6. Using factoring in solving problems, solve the following:
   a) One number is four times another. Their product is 100. What are the numbers? $x(4x) = 100$
   b) The length of a rectangle is three times the width. The area is 75 square inches. Find the dimensions of the rectangle.

   Illustrative Solution

   Let x represent the number of feet in width. Then 3x represents the length (in feet) and $x \cdot 3x$ represents the area (in square feet).

Note: By the nature of the problem, the domain of x must be restricted to positive numbers.

$$x \cdot 3x = 75$$
$$3x^2 = 75$$
$$3x^2 - 75 = 0$$

Since 3 is a non-zero number, we may divide both numbers by 3.

$$x^2 - 25 = 0$$
$$(x + 5)(x - 5) = 0$$

Then $x + 5 = 0$ or $x - 5 = 0$

and $x = -5$ and $x = 5$

Have students note that the quadratic equation has two solutions but -5 must be rejected as a solution of the equation since the domain of x is positive numbers. The solution set of the equation is 5. Have students check the solution against the conditions of the problem to see whether it is a solution to the problem.

APPLICATIONS:

Factor:

a) $3c^2 + 8c + 5$
b) $3x^2 - 14x - 5$
c) $2r^2 - r - 3$

Solve:

d) $2y^2 + y - 15 = 0$
e) $2t^2 - 9t + 4 = 0$
f) $3x^2 + 11x - 20 = 0$

g) The area of a rectangle is $2x^2 + 5x - 12$. Express the length and width as a binomial in x.

h) If one side of a square is increased by 4 inches and an adjacent side is decreased by 4 inches, the area of the resulting rectangle is 20 square inches. Find one side of the square.

i) The altitude of a parallelogram is 3 units smaller than the base. The area is 54 square units. Find the base and the altitude. (The negative solution has no meaning.)

j) The square of a certain integer is 5 more than the next consecutive integer. Find the number.

Solve:

k) $25x^2 = 100$

l) $12x^2 = 8x$

m) $9x^2 - 12x + 4 = 0$

SUMMARY:

1. If the factors of $kx^2 - 13x - 5$ are $(2x + p)$ and $(3x + 1)$, what are the values of k and p?
2. Is it possible for a quadratic equation to have no real roots? Only one distinct root ? (Have students explain and provide an example.)

HOMEWORK:

1. A window screen is 10 inches longer than it is wide. Its area is 375 square inches. Find its dimensions.
2. A rectangle has a perimeter of 20 inches and an area of 24 square inches. Find its dimensions.
3. John wishes to double the area of his garden by increasing the length and the width by the same amount. If the original garden is 18 feet long and 12 feet wide, by how many feet must each dimension be increased?
4. Solve:

$2y^2 = 24y - 72$

$2a^2 + 3a = -1$

$2x^2 + 5x - 3 = 0$

$4x^2 - 12x + 9 = 0$

$x^2 - 20x + 100 = 0$

$2a^2 + 9a - 5 = 0$

$2t^2 + 7t - 15 = 0$

$3x^2 + 14x - 5 = 0$

$4x^2 + 4x + 1 = 0$

$x^2 - 10x + 25 = 0$

Lesson: 33

AIM: How do we graph a quadratic relation on a set of coordinate axes?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Graph a line.
2. Identify the equation of a parabola.
3. Graph a parabola using a table.
4. Write the equation of the axis of symmetry of a parabola.
5. Identify the turning point of a parabola.
6. Determine under which circumstances a parabola opens upward or downward.

VOCABULARY: parabola, turning point, vertex, axis of symmetry

MOTIVATION:

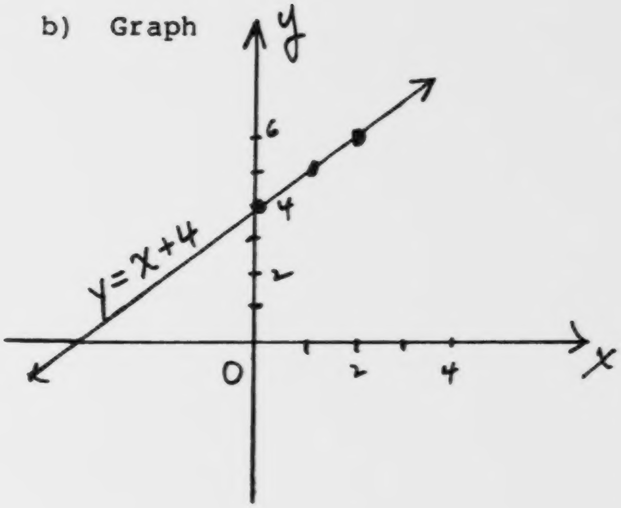
Graph $y = x + 4$.

DEVELOPMENT:

1. Refer to motivation.
   a) Make a table of values

x	$x + 4$	y
0	0 + 4	4
1	1 + 4	5
2	2 + 4	6

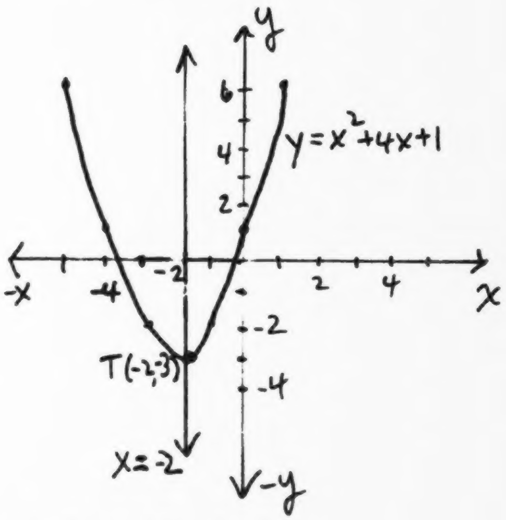
   b) Graph



   c) Label the graph.

2. Consider Graph $y = x^2 + 4x + 1$ where $-5 \leq x \leq 1$.

x	$x^2 + 4x + 1$	y
-5	25 - 20 + 1	6
-4	16 - 16 + 1	1
-3	9 - 12 + 1	-2
-2	4 - 8 + 1	-3
-1	1 - 4 + 1	-2
0	0 + 0 + 1	1
1	1 + 4 + 1	6



When plotting the points for the function $y = x^2 + 4x + 1$, the students should recognize that the slope between each pair of points is different. This property will lead them to a function different from the linear function $y = x + 4$ whose slope is constant. We call a function of the form $y = ax^2 + bx + c$ ($a \neq 0$) a quadratic function. Connecting representative points with line segments as the graph for this function is an error.

By introducing the fractional values in the domain, and connecting the points in order of increasing abscissa value, the pattern for the graph will be more clearly visible. It is a smooth curve. The graph should be identified as a parabola and drawn. The graph of the points in the solution set of any quadratic function will be a parabola.

3. The parabola is symmetric with respect to a line of reflection, or an axis of symmetry, whose equation is $x = -2$. Just like a "mirror image," every point on the parabola to the left of $x = -2$ matches a point on the parabola to the right of $x = -2$ and vice versa.
The equation of the axis of symmetry of the parabola $y = ax^2 + bx + c$ is $x = \frac{-b}{2a}$.
e.g.,

$$y = x^2 + 4x + 1 \qquad a = 1,\ b = 4,\ c = 1$$

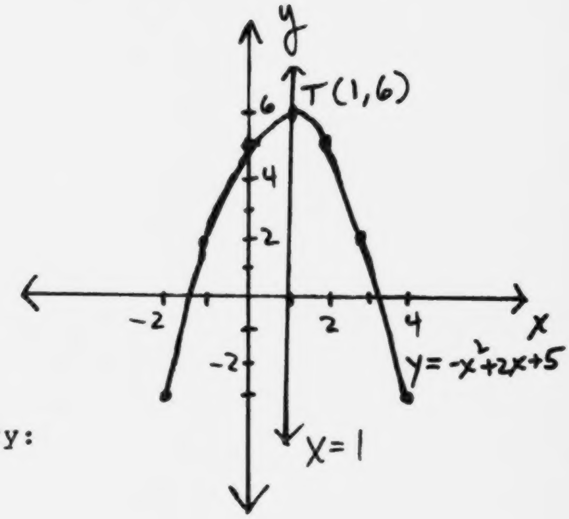
$$x = \frac{-(4)}{2(1)} = \frac{-4}{2} = -2$$

3. The graph of the parabola $y = x^2 + 4x + 1$ opens upward and contains a minimum point $(-2,-3)$ located on its axis

of symmetry. This point is called the minimum because it contains the smallest value of y for the graph. Notice that as x increases from - 5 to 1, y decreases until the minimum point is reached, then "turns," and y starts to increase. For this reason, (-2,-3) is also called the turning point (or vertex) of this parabola.

5. Graph the parabola $y = -x^2 + 2x + 5$ where $-2 \leq x \leq 4$. Find axis of symmetry. Find the coordinates of the turning point.

x	$-x^2 + 2x + 5$	y
-2	$-(-2)^2 + 2(-2) + 5$	-3
-1	$-(-1)^2 + 2(-1) + 5$	2
0	$-(0)^2 + 2(0) + 5$	5
1	$-(1)^2 + 2(1) + 5$	6
2	$-(2)^2 + 2(2) + 5$	5
3	$-(3)^2 + 2(3) + 5$	2
4	$-(4)^2 + 2(4) + 5$	-3



Equation of the axis of symmetry:

$$x = \frac{-b}{2a} = \frac{-(2)}{2(-1)} = \frac{-2}{-2} = 1$$

$x = 1$

6. Compare the above two parabolas. What conclusions can you draw?

Rule 1: If $y = ax^2 + bx + c$ and a is positive, $a > 0$, the graph of the equation is a parabola that opens upward and has a minimum turning point.

Rule 2: If $y = ax^2 + bx + c$ and a is negative, $a < 0$, the graph of the equation is a parabola that opens downward and has a maximum turning point.

APPLICATIONS:

1. Graph $y = x^2 - 4x + 3$ where $-1 \leq x \leq 5$. Find the equation of the axis of symmetry. Find the coordinates of the turning point.

2. Graph $y = -x^2 + 4x - 3$ where $-1 \leq x \leq 5$. Find the equation of the axis of symmetry. Find the coordinates of the turning point.

3. Find the equation of the axis of symmetry of each of the following:

   (a) $y = \frac{x^2}{3} - 6x + 5$ (b) $y = x^2 + x^2 - 2x + 3$

4. Find the coordinates of the turning point. State whether turning point is a maximum or a minimum.

   (a) $y = x^2 - 6x + 5$ (b) $y = 2x^2 + 5x + 2$

5. Without graphing find the coordinates of the x and y intercepts of the graph of $y = 2x^2 - 3x - 5$.

SUMMARY:

1. The parabola whose equation is $y = ax^2$ passes through the point (2,3). Find the value of a.
2. The graph of the equation $y = x^2 + 3x + k$ passes through the point (2,0). Find the value of k.
3. The graph of the equation $x^2 + bx + c = y$

   a) always b) sometimes c) never

   passes through the origin.
4. As the value of a in the equation $y = ax^2$ $(a>0)$ increases, how does the shape of its graph change?

HOMEWORK:

1. Give the equation of the axis of symmetry of the graph of the function.

   a. $y = x^2 - 16$ d. $y = x^2 - 16$

   b. $y = x^2 + 8x + 7$ e. $y = 2x^2 + 5x + 2$

   c. $y = 3x^2 - 6x$

2. Find the coordinates of the turning point and draw the graph of the function.

   a. $y = 4x_2$ c. $y = x^2 - 3x$

   b. $y = x^2 - 2x - 8$ d. $y = x^2 - x + 6$

3. For each of the following functions

   a. Draw a graph using the indicated values of x.

   b. Mark the turning point with the letter T and state whether it is a minimum or maximum point.

   c. Write the coordinates of the turning point.

d. Draw the axis of symmetry and state its equation.

(1) $y = x^2$ from $x = -5$ to $x = 5$
(2) $y = -2x^2$ from $x = -2$ to $x = 2$
(3) $y = -x^2 + 4$ from $x = -2$ to $x = 6$
(4) $y = x^2 - 4x$ from $x = -2$ to $x = 6$
(5) $y = -x^2 + 6x - 8$ from $x = 0$ to $x = 6$

Lesson: 34

AIM: How do we use graphing to solve quadratic equations? (Part I)

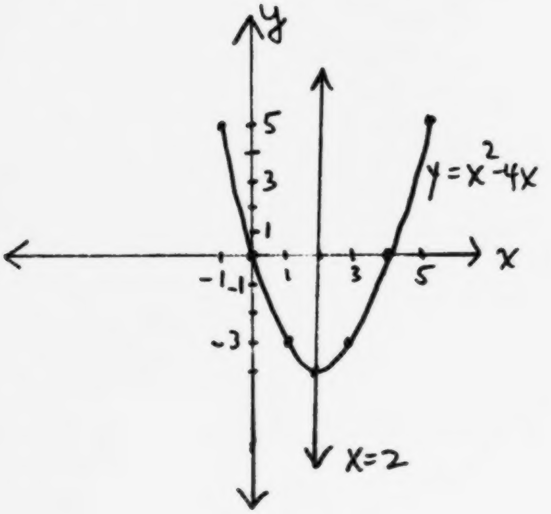
PERFORMANCE OBJECTIVES: The student will be able to...

1. Determine the associated parabola of a quadratic equation.
2. Graph the associated parabola.
3. Determine the roots of the quadratic equation from the graph of the associated parabola.

VOCABULARY: no new terms

MOTIVATION:

Draw the graph of the equation $y = x^2 - 4x$ in the interval $-1 \le x \le 5$.



x	$x^2 - 4x$	y
-1	$(-1)^2 - 4(-1)$	5
0	0	0
1	1 - 4	-3
2	$2^2 - 4(2)$	-4
3	$3^2 - 4(3)$	-3
4	$4^2 - 4(4)$	0
5	$5^2 - 4(5)$	5

$$x = \frac{-b}{2a}$$

$$x = \frac{-(-4)}{2(1)} = \frac{4}{2} = 2$$

DEVELOPMENT:

1. Refer to motivation.

(a) Find the roots of the quadratic equation $x^2 - 4x = 0$.

METHOD I: FACTORING

$$x^2 - 4x = 0$$
$$x(x - 4) = 0$$
$$x = 0 \quad \Big| \quad x - 4 = 0$$
$$x = 4$$

Ans: $x = 0$, $x = 4$

METHOD II: GRAPHIC SOLUTION

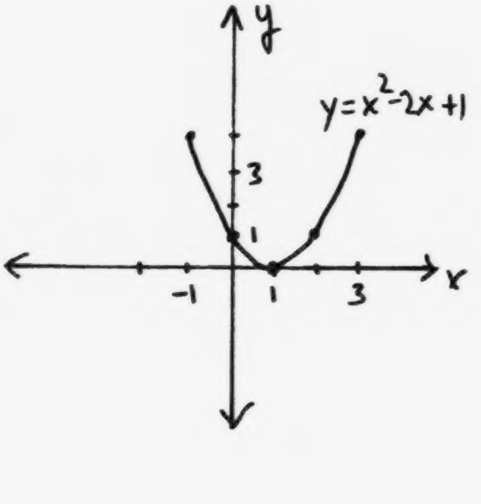
Notice that the graph of the parabola intersects the x-axis at two points: $x = 0$ and $x = 4$. These values, 0 and 4, are called the x-intercepts of the parabola, and they are the roots of the quadratic equation $x^2 - 4x = 0$. Why is this true?

The equation of the parabola is $y = x^2 - 4x$. The equation of the x-axis is $y = 0$.

Since the parabola intersects the x-axis only at points where $y = 0$, then only the values $x = 0$ and $x = 4$ will make the equation $x^2 - 4x = 0$ true.

(b) Using the graph drawn in part (a), find the roots of the quadratic equation $x^2 - 4x = 5$.

2. Draw the graph of the equation $y = x^2 - 2x + 1$ in the interval $(-1 \leq x \leq 3)$.



x	$x^2 - 2x + 1$	y
-1	$(-1)^2-2(-1)+1$	4
0	$0^2-2(0)+1$	1
1	$1^2-2(1)+1$	0
2	$2^2-2(2)+1$	1
3	$3^2-2(3)+1$	4

One point of intersection has two real and equal roots, $x = 1$ or $x = 1$.

(a) Find the roots of the quadratic equation $x^2 - 2x + 1 = 0$.

METHOD I: FACTORING

$$x^2 - 2x + 1 = 0$$

$$(x - 1)(x - 1) = 0$$

$$x - 1 = 0 \quad \Big| \quad x - 1 = 0$$

$$x = 1 \quad \Big| \quad x = 1$$

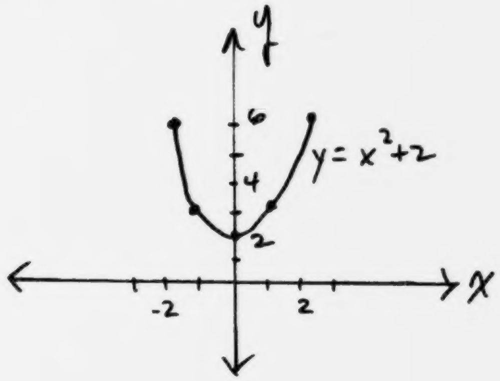
METHOD II: GRAPHIC SOLUTION

The parabola intersects the x-axis at only one point. The solution of the quadratic equation is the single x-intercept, $x = 1$.

The quadratic equation $x^2 - 2x + 1$ has two real and equal roots.

(b) Find the roots of $x^2 - 2x + 1 = k$, where $k = 4$

3.(a) Draw the graph of the equation $y = x^2 + 2$ in the interval $(-2 \leq x \leq 2)$.



x	$x^2 + 2$	y
-2	$(-2)^2+2$	6
-1	$(-1)^2+2$	3
0	0^2+2	2
1	1^2+2	3
2	2^2+2	6

Find the roots of the quadratic equation $x^2 + 2 = 0$.

METHOD I: ALGEBRAIC SOLUTION

$$x^2 + 2 = 0$$

$$x^2 = -2 \qquad \text{no real roots}$$

METHOD II: GRAPHIC SOLUTION

The parabola $y = x^2 + 2$ does not intersect the x-axis. Therefore there is no real value of x for which $x^2 + 4 = 0$.

(b) What is the smallest value of k for which the roots of $x^2 + 4 = k$ are real?

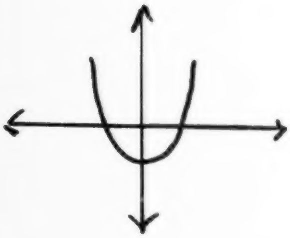
APPLICATIONS:

1. Graph $y = x^2 + 2x + 3$ in the interval $-4 \le x \le 2$.
   Find the roots of $x^2 + 2x + 3 = 0$
2. Graph $y = x^2 - 2x + 2$ in the interval $-2 \le x \le 4$.
   Find the roots of $x^2 - 2x + 2 = 0$

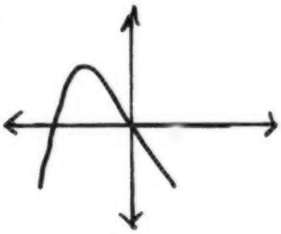
SUMMARY:

The graph of $y = ax^2 + bx + c$ has a maximum turning point and the equation $ax^2 + bx + c = 0$ has two equal roots. Which of the following may represent the graph of $y = ax^2 + bx + c$?

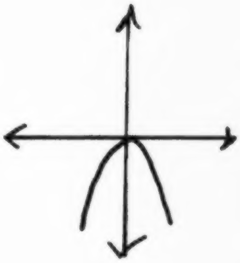
a)



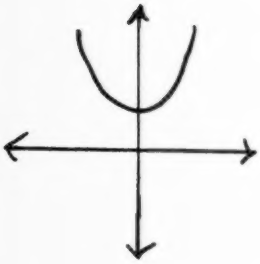
b)



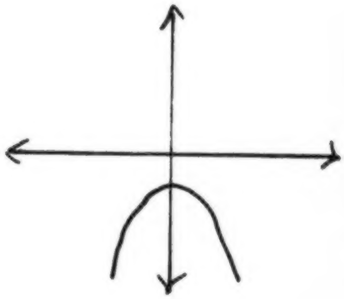
c)



d)



e)



HOMEWORK:

1. Graph $y = x^2 - 4$ in interval $-3 \le x \le 3$.
   Find the roots of the equation $x^2 - 4 = 0$.
2. Graph $y = x^2 + 4$ in interval $-3 \le x \le 3$.
   Find the roots of the equation $x^2 + 4 = 0$.
3. Graph $y = x^2 - 4x + 4$ in interval $-1 \le x \le 5$.
   Find the roots of the equation $x^2 - 4x + 4 = 0$.

Lesson: 35

AIM: How do we use graphing to solve quadratic equations? (Part II)

PERFORMANCE OBJECTIVES: The student will be able to...

1. Determine the roots of the quadratic equation from the graph of the associated parabola.
2. Use the graph of the associated parabola to determine when a quadratic equation does not have real roots, has two real and unequal roots, or has two real and equal roots.

VOCABULARY: No new terms

MOTIVATION:

The following parabolas intersect the x axis at how many points?

a) $y = x^2 - 4$

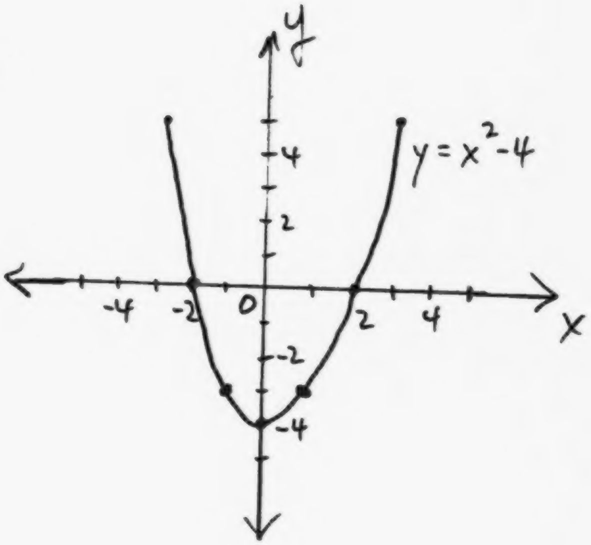
$$x^2 - 4 = 0$$

$$(x+2)(x-2) = 0$$

$$x + 2 = 0 \quad \Big| \quad x - 2 = 0$$

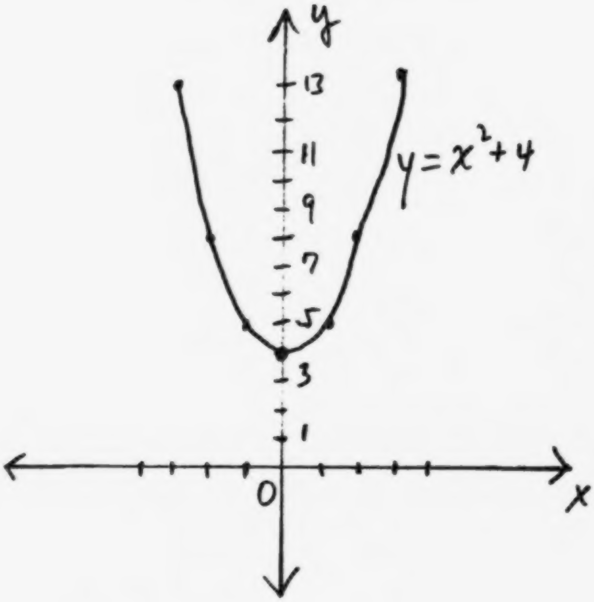
$$x = -2 \quad \Big| \quad x = 2$$

x	$x^2 - 4$	y
-3	$(-3)^2-4$	5
-2	$(-2)^2-4$	0
-1	$(-1)^2-4$	-3
0	$0-4$	-4
1	1^2-4	-3
2	2^2-4	0
3	3^2-4	5



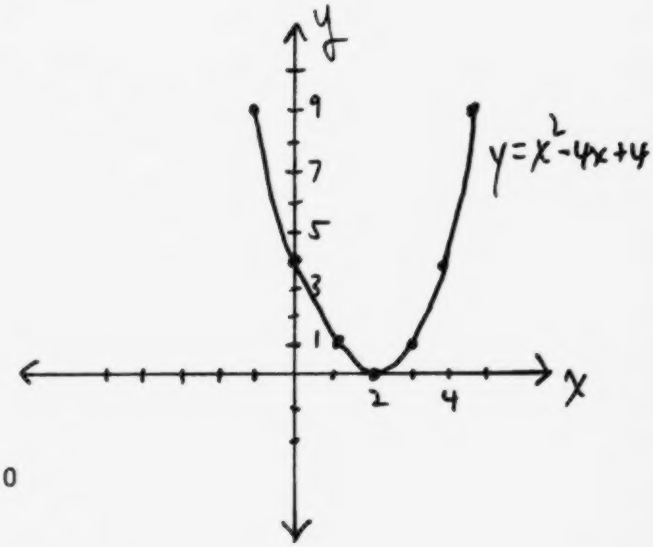
b) $y = x^2 + 4$

x	$x^2 + 4$	y
-3	$(-3)^2+4$	13
-2	$(-2)^2+4$	8
-1	$(-1)^2+4$	5
0	$0+4$	4
1	1^2+4	5
2	2^2+4	8
3	3^2+4	13



c) $y = x^2 - 4x + 4$

x	$x^2 -4x + 4$	y
-1	$(-1)^2-4(-1)+4$	9
0	$0-4(0)+4$	4
1	$1^2-4(1)+4$	1
2	$2^2-4(2)+4$	0
3	$3^2-4(3)+4$	1
4	$4^2-4(4)+4$	4
5	$5^2-4(5)+4$	9



$$x^2 - 4x + 4 = 0$$

$$(x - 2)(x - 2) = 0$$

$$\begin{array}{r|l} x - 2 = 0 & x - 2 = 0 \\ x = 2 & x = 2 \end{array}$$

DEVELOPMENT:

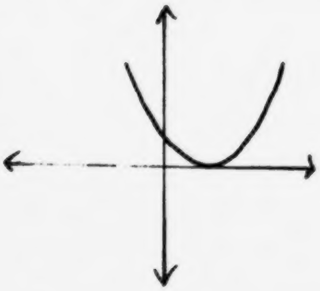
1. Refer to motivation: Reviewing yesterday's homework assignment.

   Elicit from the students the information needed for the following chart.

Parabola	Intersections with x axis	Nature of root	Quadratic Equation
$y = x^2 - 4$	2	real, unequal	$x^2 - 4 = 0$
$y = x^2 + 4$	0	not real	$x^2 + 4 = 0$
$y = x^2 - 4x + 4$	1	real, equal	$x^2 - 4x + 4 = 0$

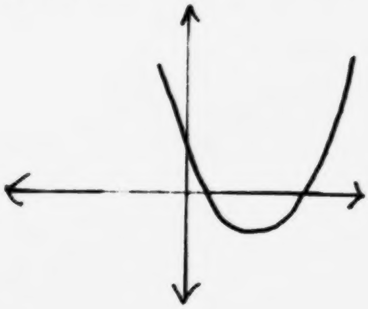
Cases:

1 pt. of intersection



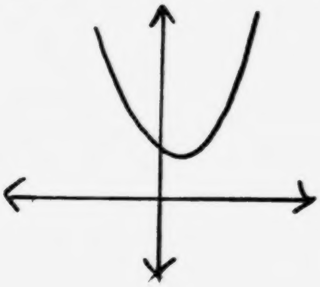
$ax^2 + bx + c = 0$ has 2 real and = roots

2 pts. of intersection



$ax^2 + bx + c = 0$ has 2 real and unequal roots

0 pts. of intersection



$ax^2 + bx + c = 0$ has no real roots

APPLICATIONS:

1. Indicate the number of points that the parabola $y = ax^2 + bx + c$ and the x-axis have in common if the roots of $ax^2 + bx + c = 0$ are:
   - a) 2 and -2
   - b) $3 + \sqrt{2}$ and $3 - \sqrt{2}$
   - c) 12 and 12
   - d) $3 + \sqrt{-2}$ and $3 - \sqrt{-2}$
   - e) 0 and - 7
2. Graph and determine nature of roots.
   - (a) $y = -x^2 + 2x$
   - (b) $y = x^2 - 2x + 1$
3. (a) Solve the quadratic equation $x^2 - 2x - 3 = 0$ graphically.

(b) What is the smallest integral value of k for which the roots of $x^2 - 2x - 3 = k$ are not real? Are real and equal?

SUMMARY:

Describe the nature of the roots of $ax^2 + bx + c = 0$ if:

- a) the parabola $y = ax^2 + bx + c$ intersects the x-axis in two points.
- b) the parabola $y = ax^2 + bx + c$ is tangent to the x-axis.
- c) the parabola $y = ax^2 + bx + c$ does not intersect the x-axis.

HOMEWORK:

1. Given the parabola $y = x^2 - 5x + 4$:
   - a) Write the equation of its axis of symmetry.
   - b) Write the coordinates of the turning point of the parabola.
   - c) Graph $y = x^2 - 5x + 4$ from $x = 0$ to $x = 5$, including the turning point.
   - d) Find the roots of the equation $x^2 - 5x + 4 = 0$.
2. a) Graph the equation $y = -x^2 + 6x - 4$ from $x = 0$ to $x = 6$ inclusive.
   - b) For $-x^2 - 6x - 4 = 0$, is it true that
     - 1) the roots are real?
     - 2) the roots are equal?
     - 3) the roots are integers?
     - 4) there are no real roots?
3. a) Graph $y = x^2 - 6x + 10$. $(0 \leq x \leq 6)$
   - b) Solve $x^2 - 6x + 10 = 0$.
4. a) Graph $y = 2x^2 + 8x + 6$. $(-4 \leq x \leq 0)$
   - b) Solve $2x^2 + 8x + 6 = 0$.

Lesson: 36

AIM: How do we use the "completing the square" technique to solve quadratic equations?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Review the definition of a perfect square trinomial.
2. Find the constant term of a perfect square trinomial.
3. Transform a quadratic equation into an equation with a perfect square trinomial.
4. Use the "completing the square" technique to solve a quadratic equation which cannot be solved by factoring.
5. Express each root of the quadratic equation in simplest radical form.

VOCABULARY: no new terms

MOTIVATION: Solve:

1) $x^2 = 4$.
2) $(x - 1)^2 = 4$.
3) $(x + 2)^2 = 3$.

DEVELOPMENT:

1. Refer to motivation.
   1) $x^2 = 4$ Ans: $x = \pm 2$
   2) $(x - 1)^2 = 4$
   $x - 1 = 2$ $x - 1 = -2$ Ans: $x = 3, x = -1$
   3) $(x + 2)^2 = 3$
   $x + 2 = \sqrt{3}$ $x + 2 = -\sqrt{3}$ Ans: $x = \sqrt{3} - 2, x = -\sqrt{3} - 2$
2. Consider $x^2 + 6x = 8$. How can we solve such an equation which cannot be factored and is not a perfect square? In $x^2 + 6x + ?$, ask the class if they can "guess" the missing term which would make the left member a perfect square trinomial.

$$x^2 + 6x + 9 = 8 + 9$$
$$(x + 3)^2 = 17$$
$$x + 3 = \pm\sqrt{17}.$$

If the polynomial $ax^2 + bx + c$ (or some multiple of it) is not factorable over the set of integers, then the roots of $ax^2 + bx + c = 0$ are not rational and other techniques are needed to find them. One such technique is the method of completing the square. Completing the square is based upon building a perfect square trinomial for one member of the equation and then taking square root of both members of the equation.

3. Solve the quadratic equation $x^2 - 6x - 1 = 0$.

$$x^2 - 6x - 1 = 0$$
$$x^2 - 6x = 1$$
$$x^2 - 6x + 9 = 1 + 9$$
$$(x - 3)^2 = 10$$
$$x - 3 = \pm\sqrt{10}$$
$$x = 3 \pm\sqrt{10}$$

4. Elicit the relationship that when the coefficient of x^2 is 1, the constant term is the square of one-half the x-coefficient. If $x^2 + bx + c$ is to be a perfect square, $c = (b/2)^2$. Can we show that this will always be the case? Square $(x + a)$ to show this.

5. Now extend the technique to cases such as $2x^2 - 5x + 1 = 0$ where the coefficient of x^2 is not 1. Dividing by 2 will transform the equation to the previous type and will permit application to equations with roots which are rational numbers.

Solve $2x^2 - 5x - 1 = 0$.

$$2x^2 - 5x = 1$$
$$x^2 - \frac{5x}{2} = \frac{1}{2}$$
$$x^2 = \frac{5x}{2} + \frac{25}{16} = \frac{1}{2} + \frac{25}{16}$$
$$(x - \frac{5}{4})^2 = \frac{33}{16}$$
$$x - \frac{5}{4} = \pm\frac{\sqrt{33}}{4}$$
$$x = \frac{5 \pm\sqrt{33}}{4}$$

APPLICATIONS:

1. Which of the following are trinomial squares? Explain.
   - a) $x^2 + 6x + 9$
   - b) $x^2 - 8x + 16$
   - c) $a^2 + 4a + 2$
   - d) $25y^2 + 10y + 1$
   - e) $b^2 - 2b - 1$
2. Solve the following equations by the method of completing the square.
   - a) $t^2 + t = 12$
   - b) $2x^2 - 8x = 24$
   - c) $x^2 = 10x + 144$

SUMMARY:

What term must be added to each side of the equation $ax^2 + bx + c = 0$ in order to obtain a perfect square trinomial?

HOMEWORK:

1. What number should be added to each side of the following equations to make a trinomial square? Express the resulting trinomial as the square of a binomial.
   - a) $x^2 - 2x = 0$
   - b) $b^2 + 6b = 0$
   - c) $y^2 + 12y = 0$
   - d) $a^2 - 10a = 0$
   - e) $y^2 - 20y = 0$
   - f) $t^2 + 3t = 0$
   - g) $x^2 - x = 0$
   - h) $r^2 + \frac{1}{2}r = 0$
2. Solve the following equations by the method of completing the square.
   - a) $y^2 - 6y = 7$
   - b) $a^2 - 4a - 32 = 0$
   - c) $x^2 + 2x - 48 = 0$

Lesson: 37

AIM: How do we derive and use the quadratic formula?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Apply the "completing the square" technique to derive the quadratic formula.
2. State the quadratic formula.
3. Solve quadratic equations using the quadratic formula.
4. Apply the quadratic formula to solve verbal problems which lead to quadratic equations.

VOCABULARY: No new terms

MOTIVATION:

The trinomial $x^2 + 6x + 9$ may be written as $(x + 3)^2$. What must be the value of the constant c in $x^2 + 10x + c$ in order to be able to express this trinomial as a perfect square? What is the relationship between this value and the coefficient of x?

DEVELOPMENT

1. Refer to motivation.
   If we apply the steps for completing the square to the general equation $ax^2 + bx + c = 0$, a formula will result.

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

$$ax^2 + bx = -c$$

$$x^2 + \frac{bx}{a} = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$(x + \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{4ac}{4ac}$$

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = - \frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

2. Given any quadratic equation of the form $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$, we can find the roots of the equation by the quadratic formula:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

3. Solve the equation by using the quadratic formula.

   a) $x^2 - 9x + 10 = 0$ $\quad$ $a = 1,\ b = -9,\ c = 10$

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   $$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(10)}}{2(1)}$$

   $$x = \frac{9 \pm \sqrt{81 - 40}}{2}$$

   $$x = \frac{9 \pm \sqrt{41}}{2}$$

   b) $3x^2 - 2x - 5 = 0$ $\quad$ $a = 3,\ b = -2,\ c = 5$

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{(-2) \pm \sqrt{(-2)^2 - 4(3)(-5)}}{2(3)}$$

$$x = \frac{2 \pm \sqrt{4 + 60}}{6}$$

$$x = \frac{2 \pm \sqrt{64}}{6}$$

$$x = \frac{2+8}{6} = \frac{10}{6} = \frac{5}{3}, \qquad x = \frac{2-8}{6} = \frac{-6}{6} = -1$$

c) Solve $9x^2 - 6x + 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad a = 9,\ b = -6,\ c = 1$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(9)(1)}}{2(9)}$$

$$x = \frac{6 \pm \sqrt{36 - 36}}{18} = \frac{6}{18} = \frac{1}{3}$$

Ans: $x = \frac{1}{3}, \frac{1}{3}$

4. In a rectangle, the length is twice the width and the area of the rectangle is 6. Find the dimensions of the rectangle.

Let x = width, 2x = length

Area = length x width

$$6 = (2x)(x)$$

$$6 = 2x^2$$

$$0 = 2x^2 - 6$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-0 \pm \sqrt{0^2 - 4(2)(-6)}}{2(2)}$$

$$x = \frac{\pm\sqrt{48}}{4} = \frac{\pm 16\sqrt{3}}{4}$$

$$x = \pm \frac{4\sqrt{3}}{4} = \pm\sqrt{3}$$

We reject $x = -\sqrt{3}$ because in the verbal problem we are interested in positive numbers.
Width of the rectangle is $\sqrt{3}$. Length of the rectangle is $2(\sqrt{3}) = 2\sqrt{3}$.

APPLICATIONS:

1. What is the positive root of the equation $x^2 + 2x - 15 = 0$?

2. What are the roots of the equation $x^2 - 4x - 3 = 0$?

   a) $\frac{-4 \pm \sqrt{28}}{2}$ b) $\frac{4 \pm \sqrt{28}}{2}$ c) $\frac{1}{2}, \frac{5}{2}$ d) 1,5

3. The roots of the equation $2x^2 - 6x + 3 = 0$ are

   a) $\frac{-6 \pm \sqrt{12}}{4}$ c) $\frac{6 \pm \sqrt{12}}{4}$

   b) $\frac{-6 \pm \sqrt{60}}{4}$ d) $\frac{6 \pm \sqrt{60}}{4}$

4. Solve and check:
   (a) $x^2 + 3x - 40 = 0$
   (b) $x^2 + 16 = 8x$
   (c) $3x^2 + 5x = -2$

5. The base of a rectangle measures 1 foot less than twice its height. If the area is 45 square feet, find the dimensions of the rectangle.

SUMMARY:

Explain the error and correct the solution in each of the following.

a) A student simplified $\frac{4 \pm \sqrt{4}}{2}$, by writing: $2 \pm \sqrt{4}$

b) A student simplified $\frac{10 \pm \sqrt{18}}{2}$, by writing: $5 \pm \sqrt{9}$

c) From $x^2 - 5x + 6 = 0$, a student got $x = \frac{-5 \pm \sqrt{24-24}}{2}$

HOMEWORK:

1. Solve the following by using the quadratic formula. Express irrational roots in simplest radical form.

   $2x^2 + 2 = 5x$

   $3x^2 - 3x - 2 = 0$

   $2(x^2 - 3) = x$

2. The length of a rectangle is 3 feet longer than twice its width. If the area is 27 sq. ft., find the dimensions of the rectangle.

3. If 4 times a certain number is subtracted from the square of the number, the result is 25. Find the number that satisfies this condition.

4. There are two consecutive positive integers such that the square of the first decreased by twice the second is 33. What are these integers?

Lesson: 38

AIM: How can the quadratic formula be used to determine the nature of the roots of a quadratic equation?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define the discriminant of the quadratic formula.
2. Use the discriminant to determine when the roots of a quadratic equation are rational and unequal, irrational and unequal, rational and equal, or not real.

VOCABULARY: discriminant

MOTIVATION:

By the quadratic formula, the two roots of a quadratic may be expressed as:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

What part of the formula determines whether x_1 and x_2 will be real or imaginary?
Identify $b^2 - 4ac$ as the discriminant. The value of the discriminant can help us predict the type of roots.

DEVELOPMENT:

Consider yesterday's lesson.
When we used the quadratic formula to solve $x^2 - 9x + 10 = 0$ we obtained:

$$x_1 = \frac{9 + \sqrt{41}}{2} \quad \text{and} \quad x_2 = \frac{9 - \sqrt{41}}{2}$$

The value of the discriminant $b^2 - 4ac$ is 41 (not a perfect square). The nature of the roots are irrational and unequal.

When we used the quadratic formula to solve $3x^2 - 2x - 5 = 0$ we obtained::

$$x_1 = \frac{5}{3} \quad \text{and} \quad x_2 = -1$$

The value of the discriminant $b^2 - 4ac$ is 64 (a perfect square). The nature of the roots are rational and unequal.

When we use the quadratic formula to solve $9x^2 - 6x + 1 = 0$:

$$x_2 = \frac{1}{3} \quad \text{and} \quad x_2 = \frac{1}{3}$$

The value of the discriminant $b^2 - 4ac$ is 0. The nature of the roots are rational and equal.

Return to motivation. Consider the quadratic equation:

$$2x^2 - 6x + 5 = 0 \qquad a = 2,\ b = -6,\ c = 5$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(5)}}{2(2)}$$

$$x = \frac{6 \pm \sqrt{36-40}}{4} = \frac{6 \pm \sqrt{-4}}{4}$$

$$x_1 = \frac{6 + \sqrt{-4}}{4} = \frac{6 + 2i}{4} = \frac{\cancel{2}(3 + i)}{\cancel{2}\cdot 2} = \frac{3 + i}{2}$$

$$x_2 = \frac{6 - \sqrt{-4}}{4} = \frac{6 - 2i}{4} = \frac{\cancel{2}(3 - i)}{\cancel{2}\cdot 2} = \frac{3 - i}{2}$$

The value of the discriminant $b^2 - 4ac$ is -4 (negative). The nature of the roots are imaginary (no real roots.)

Have the class build a table showing the various values of the discriminant and the associated classification of the roots.

If the discriminant is	Roots are
0	Real, rational and equal
A perfect square $\neq 0$	Real, rational and unequal (Have the class note the equation is factorable in the above two cases because the roots are rational.)
A positive number but not a perfect square	Real, irrational and unequal
A negative number	Imaginary and unequal

With the help of the discriminant, we can answer the following questions about the nature of the roots: 1) Are they real or imaginary? 2) If they are real, are they rational or irrational? 3) If they are rational, are they equal or unequal?

Stress that the statements in the table are true provided the coefficients of the equation are rational numbers. Note, for example, that $x^2 + 2\sqrt{3}x + 3 = 0$ has two irrational roots although the discriminant is 0.

APPLICATION:

1. Compute the discriminant in each of the following:

   (a) $x^2 - x + 1 = 0$ (c) $x^2 + 6x = 8$

   (b) $-5x^2 - 4x + 13 = 0$ (d) $x^2 = 4$

2. Describe the nature of the roots of the above.

3. State the nature of the roots of the discriminant is 16, -4,-49,17,86, 0.

SUMMARY:

It is known that one of the numbers, 2, $\frac{2}{3}$, $\frac{2+i\sqrt{2}}{3}$, $2 - \sqrt{2}$ is a root of $3x^2 - 4x + 2 = 0$. Can you tell which is the root without solving the equation and without substituting?

HOMEWORK:

1. a) Solve the equation $x^2 - 3x - 4 = 0$ by
      a. factoring
      b. graphing
      c. completing the square
      d. the quadratic formula

   b) What is the nature of the roots? At how many points does its graph intersect the x-axis? ($x^2 - 3x - 4 = y$)

2. Describe the nature of the roots of each of the following:

(a) $x^2 - 6x - 1 = 0$
(b) $2x^2 - 5x - 1 = 0$
(c) $x^2 - 4x + 4 = 0$
(d) $x^2 - 2x - 3 = 0$
(e) $x^2 - 2x + 1 = 0$
(f) $x^2 + 2x = 0$

Lesson: 39

AIM: What is the relationship between the coefficients of a quadratic equation and the sum and product of its roots?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Determine that the sum of the roots of a quadratic equation is $-\frac{b}{a}$.
2. Determine that the product of the roots of a quadratic equation is $\frac{c}{a}$.
3. Use the sum and product formulas to check the roots of quadratic equations.

VOCABULARY: sum of roots, product of roots

MOTIVATION:

Write a quadratic equation with integral coefficients whose roots are (a) 2 and 3 (b) 1/2 and 3.

DEVELOPMENT:

Refer to motivation:

$r_1 = 2,\ r_2 = -3$ or
$(x - r_1)(x - r_2) = 0$
$(x - 2)(x - (-3)) = 0$
$(x - 2)(x + 3) = 0$
$x^2 + x - 6 = 0$

$r_1 = 1/2,\ r_2 = 3$
$(x - r)\ (x - r_2) = 0$
$(x - 1/2)\ (x - 3) = 0$
$x^2 - \frac{7}{2}x + \frac{3}{2} = 0$
or multiplying by 2:
$2x^2 - 7x + 3 = 0$

Is there a relationship between the sum and product of the roots of a quadratic equation and its coefficients? Analyze by organizing data in a table:

equation	r_1	r_2	$r_1 + r_2$	$r_1 \times r_2$
(a) $x^2 + x - 6 = 0$	2	-3	-1	-6
(b) $2x^2 - 7x + 3 = 0$	1/2	3	7/2	3/2
(c) $ax^2 + bx + c = 0$			$-b/a$	c/a

Proof:

Sum of roots of $ax^2 + bx + c = 0$, $a \neq 0$. Let r_1 and r_2 represent the roots.

1. $r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$
2. $r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$
3. $r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$
4. $r_1 + r_2 = \frac{-2b}{2a}$
5. $r_1 + r_2 = \frac{b}{-a}$

PRODUCT OF THE ROOTS OF $ax^2 + bx + c = 0$, $a \neq 0$.

If the roots of the quadratic equation $ax^2 + bx + c = 0$ are represented by r_1 and r_2, derive the formula for the product of the roots $r_1 \times r_2$ in terms of a and c.

Solution:

1. $r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$
2. $r_1 \times r_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$
3. $r_1 \times r_2 = \frac{b^2 - (b^2 - 4ac)}{4a^2}$
4. $r_1 \times r_2 = \frac{b^2 - b^2 + 4ac}{4a^2}$
5. $r_1 \times r_2 = \frac{4ac}{4a^2}$
6. $r_1 \times r_2 = \frac{c}{a}$

If r_1 and r_2 represent roots of a quadratic equation, then the equation in factored form is

$(x - r_1)(x - r_2) = 0$

$$\begin{array}{l} x - r_1 \\ \underline{x - r_2} \\ x^2 - r_1 \cdot x \\ \underline{\qquad\quad - r_2 x + r_1 \cdot r_2} \\ x^2 - (r_1 + r_2)x + r_1 r_2 \end{array}$$

$x^2 - (r_1 + r_2)x + r_1 r_2 = 0$

Example: Is 3,1 the solution set of $x^2 - 4x + 3 = 0$? $a = 1$, $b = -4$, $c = 3$.

$r_1 + r_2 = 3 + 1 = 4$

$r_1 r_2 = 3(1) = 3$

But $r_1 + r_2 = \frac{-b}{a} = \frac{-(-4)}{1} = 4$

$r_1 r_2 = \frac{c}{a} = \frac{3}{1} = 3$

Therefore 3 and 1 is the solution set of $x^2 - 4x + 3 = 0$.

Example: Find the sum and product of roots of the equation $x^2 - 10x = 0$. $a = 1$, $b = -10$, $c = 0$.

$r_1 + r_2 = \frac{-b}{a} = \frac{-(-10)}{1} = 10$

$r_1 r_2 = \frac{c}{a} = \frac{0}{1} = 0$

Example: If $\frac{-1}{2}$ is one root of the equation $4x^2 + 4x + 1 = 0$, find the other root.

Divide by 4: $x^2 + x + \frac{1}{4} = 0$. $a = 1$, $b = 1$, $c = \frac{1}{4}$

$r_1 + r_2 = \frac{-b}{a}$

$r_1 + r_2 = \frac{-1}{1}$

$r_1 + r_2 = -1$

Since $r_1 = -\frac{1}{2}$, substitute:

$r_1 + r_2 = -1$

$-\frac{1}{2} + r_2 = -1$

$r_2 = -\frac{1}{2}$

APPLICATIONS:

1. Find the sum and the product of the roots of the equation.

   $4x^2 - 12x + 9 = 0$
   $x^2 - 6x + 7 = 0$
   $6x^2 = 4x + 2$

2. If one root is 3 find the other root of $x^2 - 9 = 0$.
3. Find the value of K in the equation $x^2 + 5x + K = 0$ if one root is -4.
4. Write a quadratic equation with integral coefficients if one root is 1/3 and the product of the roots is -1.

SUMMARY:

1. If 2 is a root of $x^2 - 5x + n = 0$, find n.
2. In the quadratic equation $x^2 - 5x + 2 = 0$, the sum of the roots exceeds the product of the roots by

   a) -3 b) 3 c) 7

3. If the product of the roots of a quadratic equation (b and c are not both 0) is 0, then

   (a) the roots are unequal
   (b) one of the roots is 0
   (c) the sum of the roots is not 0
   (d) all of these
   (e) none of these

HOMEWORK:

1. Form a quadratic equation with integral coefficients whose roots are (a) -6 and 1. (b) 2/3 and 1/4
2. Write a quadratic equation with integral coefficients the sum and the product of whose roots are 12 and -1.
3. Find sum and product of the roots of the equation.

   $2y^2 + 4y - 7 = 0$
   $8y = 2 - y^2$

4. If one root is $-\frac{1}{3}$, find the value of k and the other root of the equation $6x^2 - 4x - k = 0$.

Lesson: 40

AIM: How can we determine the points of intersection of a parabola and a straight line?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define a quadratic linear system.
2. Draw a parabola using the axis of symmetry and a table of values.
3. Draw a line using a table of values.
4. Draw a line using the slope intercept method.
5. Determine the coordinates of the point(s) of intersection, if any, of the two graphs.
6. Check the point(s) of intersection in the original equations.

VOCABULARY: quadratic-linear system

MOTIVATION:

If the following pairs of graphs are drawn using the same set of axes, in how many points can they intersect?
(a) two lines (b) a line and a parabola
In each case, students should be asked to sketch the graphs in order to illustrate the different possibilities. A quadratic-linear system should be defined as a pair of equations, one of which is a first degree equation and the other a second degree equation.

DEVELOPMENT:

Find graphically the points of intersection of the system $y = x + 3$ and $y = 2x^2 + 6x + 5$

1. Draw the graph of $y = x + 3$.

Table of values

x	x + 3	y
0	0 + 3	3
1	1 + 3	4
2	2 + 3	5

The linear equation $y = x + 3$ may be graphed by table of values (see above) or by slope intercept method.

$y = mx + b$	slope = 1
$y = 1x + 3$	y intercept is 3

Plot the point (0,3) on y axis. From (0,3) use the slope to plot other points, showing that y increases by 1 as x increases by 1.

2. Graph $y = 2x^2 + 6x + 5$

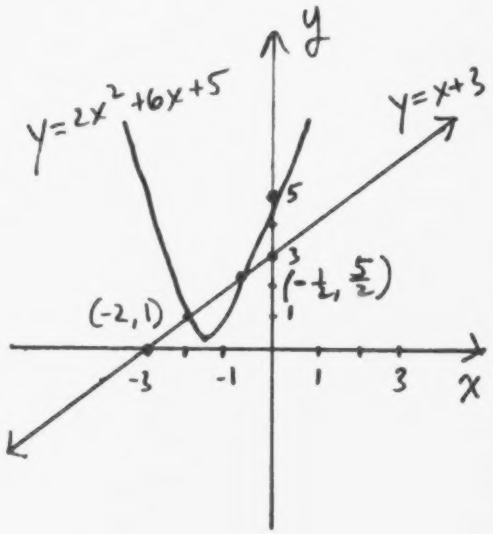
$a = 2,\ b = 6,\ c = 5$

First find the axis of symmetry.

$$x = \frac{-b}{2a} = \frac{-(6)}{2(2)} = \frac{-6}{4} = \frac{-3}{2}$$

Choose some values of x smaller than $-\frac{3}{2}$ and some values of x larger than $-\frac{3}{2}$ to develop table of values.

x	$2x^2 + 6x + 5$	y
-4	$2(-4)^2+6(-4)+5$	13
-3	$2(-3)^2+6(-3)+5$	5
-2	$s(-2)^2+6(-2)+5$	1
$-\frac{3}{2}$	$2(-\frac{3}{2})^2+6(-\frac{3}{2})+5$	$\frac{1}{2}$
-1	$2(-1)^2+6(-1)+5$	1
0	$2(0)\quad +0(0)+5$	5
1	$2(1)^2+6(1)+5$	13



3. a) On the same set of axes draw the graph of $y = 2x^2 + 6x + 5$.

   b) Determine the coordinates of the points of intersection of the graphs drawn and check these coordinates in both equations.

The graphs $y = x + 3$ and $y = 2x^2 + 6x + 5$ intersect at two points (-2,1) and $(-\frac{1}{2},\frac{5}{2})$.
Check each ordered pair in each of the original equations. Thus four checks are needed.

Check for $x = -2$, $y = 1$

$$\begin{array}{l|l} y = x + 3 & y = 2x^2 + 6x + 5 \\ +1 \overset{?}{=} -2 + 3 & 1 \overset{?}{=} 2(-2)^2 + 6(-2) + 5 \\ 1 = 1 \quad \text{True} & 1 \overset{?}{=} 8 - 12 + 5 \\ & 1 = 1 \quad \text{True} \end{array}$$

Check for $x = -\frac{1}{2}$, $y = \frac{5}{2}$.

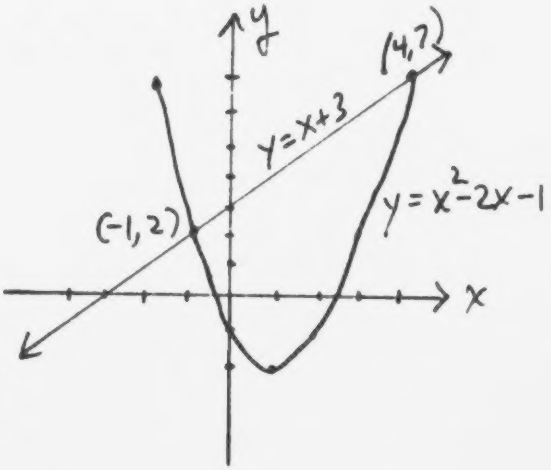
$$\begin{array}{l|l} y = x + 3 & y = 2x^2 + 6x + 5 \\ \frac{5}{2} \overset{?}{=} -\frac{1}{2} + 3 & \frac{5}{2} \overset{?}{=} 2(-\frac{1}{2})^2 + 6(-\frac{1}{2}) + 5 \\ \frac{5}{2} = \frac{5}{2} \quad \text{True} & \frac{5}{2} \overset{?}{=} \frac{1}{2} - 3 + 5 \\ & \frac{5}{2} = \frac{5}{2} \quad \text{True} \end{array}$$

Example: a) Draw the graph of the equation $y = x^2 - 2x - 1$ including all values of x such that $-2 \leq x \leq 4$.

b) On the same set of axes used in part a, draw the graph of the equation $y - x = 3$.

c) What are the coordinates of the points of intersection of the graphs in parts a and b?

x	$x^2 - 2x - 1$	y
-2	$(-2)^2 - 2(-2) - 1$	7
-1	$(-1)^2 - 2(-1) - 1$	2
0	$0^2 - 2(0) - 1$	-1
1	$1^2 - 2(1) - 1$	-2
2	$2^2 - 2(2) - 1$	-1
3	$3^2 - 2(3) - 1$	2
4	$4^2 - 2(4) - 1$	7



Solve for y

$$\begin{array}{rcl} y - x & = & 3 \\ + x & & + x \\ \hline y & = & 3 + x \end{array}$$

x	0	1	2
y	3	4	5

c) Ans: $(-1,2)$, $(4,7)$

APPLICATIONS:

For each of the following systems of equations:

a. Graph the given quadratic equation over an appropriate interval.

b. On the same set of axes, graph the linear equation.

c. Determine the coordinates of the points of intersection of the graphs in parts 1 and 2, and check these coordinates in both equations.

(1) $y = x^2 + 2x + 1$
$y = x + 3$

(2) $x^2 + 2y = 5$
$y = x + 1$

SUMMARY:

1. How can the quadratic equation $x^2 = 4x - 3$ be solved graphically as a linear quadratic system? (By graphing $y = x^2$ and $y = 4x - 3$ on the same set of axes.)
2. In how many distinct points can a line intersect a parabola?
3. Describe the roots of the quadratic equation $ax^2 = bx + c$ if the graphs of $y = ax^2$ and $y = bx + c$ do not intersect.

HOMEWORK:

Solve each quadratic linear system graphically. Show the graphs of both equations on the same axes and check.

1. $y = x^2 - 4$
$y = x + 8$

2. $x^2 + 2y = 5$
$y = x + 2$

3. $y = x^2 (-2 \leq x \leq 2)$
$y = x + 2$
$(-2 \leq x \leq 2)$
ans: $(-1,1)$ and $(2,4)$

4. $y = x^2 - 4x + 5$
$y - 2x + 4 = 0$
ans: $(3,2)$

Lesson: 41

AIM: How can we determine the points of intersection of other quadratic-linear systems?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Graph the equation of a circle whose center is not at the origin.
2. Determine the length of a radius of the circle.
3. Determine the coordinates of the points of intersection of the circle and the line, if any.
4. Check the points of intersection in the original equations.
5. Graph an equation of the form $xy = k$ using a table of values.
6. Determine the points of intersection of the graphs of $xy = k$ and a line.

VOCABULARY: equilateral (rectangular) hyperbola, circle, asymptote

MOTIVATION:

In the coordinate plane, what is the locus of all points that are 3 units from the origin? r units from the origin?

Describe the locus as a circle having the origin (0, 0) as its center and the given distance as the length of the radius. Ask students if they can define the locus using an equation.

Elicit that the graph of the equation $x^2 + y^2 = r^2$ is a circle whose center is at the origin and whose radius is r. If a circle has its center at the point (h,k) and has a radius r, the equation of the circle is

$$(x - h)^2 + (y - h)^2 = r^2$$

DEVELOPMENT:

Use the distance formula to develop the equation of a circle having a radius of r units and whose center is:

(a) at the origin (b) at the point whose coordinates are (h, k).

1. Fill in the table

equation of circle	center	radius
1. $x^2+y^2=81$	?	?
2. $(x-4)^2+(y-1)^2=100$	?	?
3. $(x+1)^2+y^2=49$	?	?
4. ?	(5,-2)	$\sqrt{13}$

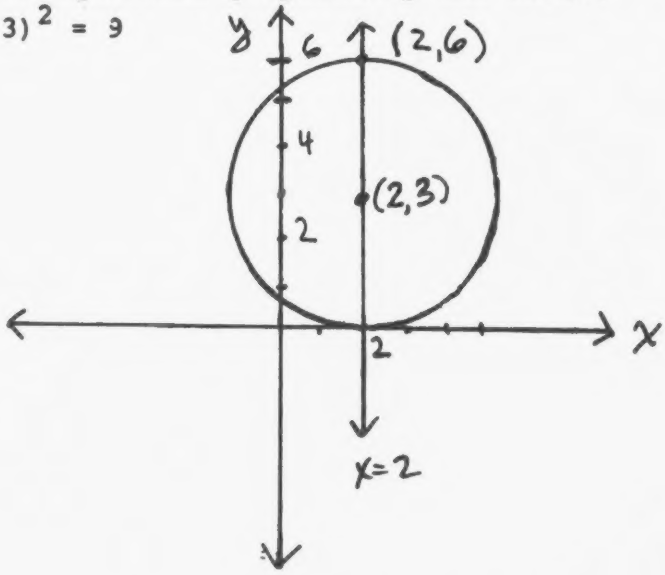
2. Solve the system of equations graphically and check.

$(x - 2)^2 + (y - 3)^2 = 9$

$x = 2$

Center (2,3)

Radius 3



Check:

$(x - 2)^2 + (y - 3)^2 = 9$

$(2 - 2)^2 + (0 - 3)^2 = 9$

$9 = 9$ True

$x = 2$

$2 = 2$ True

$(x - 2)^2 + (y - 3)^2 = 9$

$(2 - 2)^2 + (6 - 3)^2 \overset{?}{=} 9$

$3^2 \overset{?}{=} 9$

$9 = 9$ True

$x = 2$

$2 = 2$ True

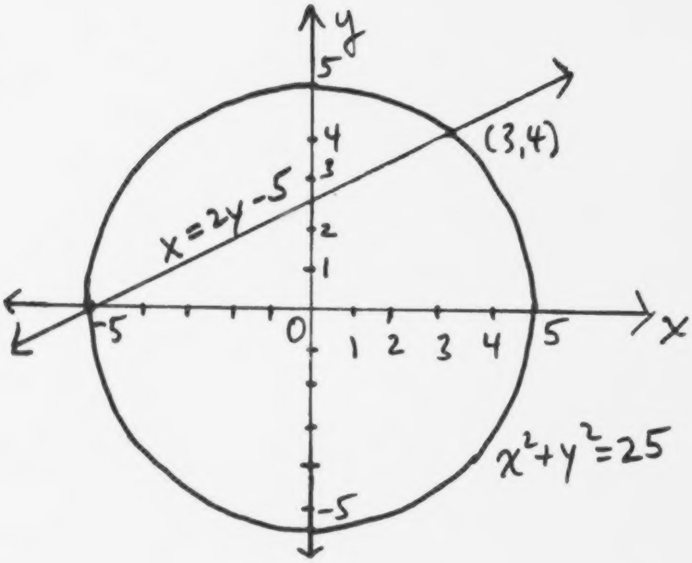
3. Solve the system of equations graphically and check.

$x^2 + y^2 = 25$

$x = 2y - 5$

SOLUTION:

Center (0,0)
Radius 5



x	-5	-3	-1
y	0	1	2

Check: $x^2 + y^2 = 25$

$(-5)^2 + 0^2 = 25$

$25 = 25$ True

$x = 2y - 5$

$-5 \overset{?}{=} 2(0) - 5$

$-5 = -5$ True

$x^2 + y^2 = 25$

$3^2 + 4^2 \overset{?}{=} 25$

$9 + 16 \overset{?}{=} 25$

$25 = 25$ True

$x = 2y - 5$

$3 \overset{?}{=} 2(4) - 5$

$3 \overset{?}{=} 8 - 5$

$3 = 3$ True

4. Consider the graph of $x^2 + y^2 = 169$. Is the point corresponding to the number pair (5,-12) on this locus? Substitute $5^2 + (-12)^2 = 169$; $25 + 144 = 169$. True.

5. Do the following points lie in the interior, on, or in the exterior of the circle whose equation is $x^2 + y^2 = 25$?

(-4,-3);	(6,1);	(5,-1);	(-2,-4).
on	exterior	exterior	interior

6. What is the radius of the circle whose equation is $x^2 + y^2 = 2$? Ans: $\sqrt{2}$.

7. **HYPERBOLA:** An equilateral hyperbola has the form $xy = k$ where $k \neq 0$.
   a) There is neither an x-intercept nor a y-intercept since, if either x or y = 0, the other variable has no value that will satisfy the equation. It is impossible to divide by 0.

b) If k is positive, the graph of xy = k will appear in the two quadrants (I and III) where x and y are both positive or both negative. If k is negative, the graph will appear in the opposite quadrants (II and IV), where x is negative and y is positive, and where x is positive and y is negative.

c) As the value of x gets larger and larger without bound, the value of y becomes smaller and smaller, but is never zero since the product of xandy is non-zero (provided $k \neq 0$). The graph of xy = k is therefore said to be asymptotic to (approaches but never intersects) the x-axis. Similarly, the graph is also asymptotic to the y-axis. Each axis is said to be an asymptote of the graph.

Example: Graph xy = 12. On the same axes, graph y = x + 1.

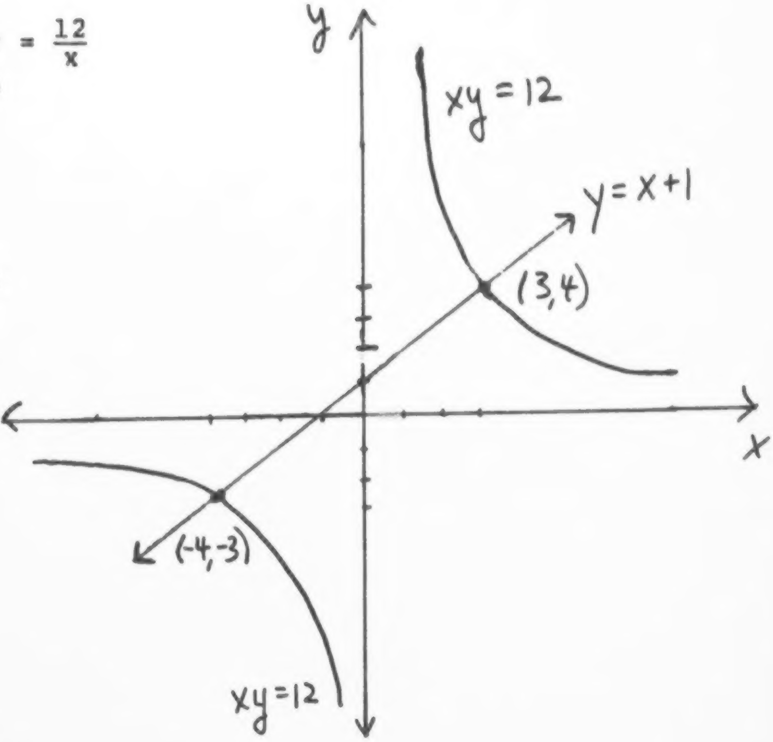
Solve for y. $y = \frac{12}{x}$

Table of Values

x	$\frac{12}{x}$	y
-12	12/-12	-1
-6	12/-6	-2
-5	12/-5	-2.4
-4	12/-4	-3
-2	12/-2	-6
-1	12/-1	-12

x	$\frac{12}{x}$	y
1	12/1	12
2	12/2	6
3	12/3	4
4	12/4	3
5	12/5	2.4
6	12/6	2
12	12/12	1

x	x + 1	y
0	0 + 1	1
1	1 + 1	2
2	2 + 1	3



Check

$(3,4)$	$(-4,-3)$
$xy = 12$	$xy = 12$
$3 \cdot 4 \stackrel{?}{=} 12$	$(-4)(-3) \stackrel{?}{=} 12$
$12 = 12$ True	$12 = 12$ True
$y = x + 1$	$y = x + 1$
$4 \stackrel{?}{=} 3 + 1$	$-3 \stackrel{?}{=} -4 + 1$
$4 = 4$ True	$-3 = -3$ True

MEDIAL SUMMARY:

Describe the method used for graphic solutions for a quadratic linear system.

a) Draw the graph of the quadratic.

b) Using the same axes, draw the graph of the linear equation.

c) Determine the coordinates of the points of intersection of the previous graphs and check these coordinates in both equations.

APPLICATIONS:

1. Write an equation of the circle whose center is the origin and which passes through each of the following points.

   a) $(3,4)$ b) $(-9,12)$ c) $(-2,-5)$ d) $(3,2)$ e) $(-5,2)$

2. Write the length of radius and the coordinates of the center of each circle whose equation is given below.

   a) $(x-5)^2 + (y-7)^2 = 64$ d) $x^2 = 19 - y^2$
   b) $(x+2)^2 + (y-3)^2 = 9$ e) $(x+3)^2 + (y-2)^2 = 36$
   c) $2x^2 + 2y^2 = 32$ f) $(y+3)^2 = 81 - x^2$

3. Solve graphically and check algebraically:

   a) $4y = 3x + 25$
   $x^2 + y^2 = 25$

   (The solution for this system is one point $(-3,4)$. The algebraic solution of these two equations will result in a quadratic equation with equal roots.)

   b) $x + y = 1$
   $xy = 12$

(The solution for this system is the empty set. The algebraic solution of these two equations will result in a quadratic equation with a negative discriminant and imaginary roots.)

SUMMARY:

1. How many points of intersection can a linear and quadratic system have?
2. Which of the following systems of equations may intersect in 3 points?
    a) $y=mx+b$, $(x-a)^2 + (y-b)^2=r^2$
    b) $y = \frac{k}{x}$, $y = mx+b$
    c) $x^2+y^2 = r^2$, $y = ax^2+bx+c$
    d) $xy = k$, $x^2+y^2 = r^2$

HOMEWORK:

1. Find the equation for each of the following circles.
    a) Center (3,-2), radius 6 units
    b) Center at origin, passing through (3,1)
    c) Center at origin, measure of radius 4 units
2. Determine the coordinates of the center and the length of the radius.
    a) $x^2 + y^2 = 36$
    b) $(x - 5)^2 + (y + 4)^2 = 18$
3. Solve graphically and check algebraically
    a) $y = 2x - 4$, $y = 3x^2 - 7x + 2$ Ans: (2,0);(1,-2)
    b) $x^2 + y^2 = 25$, $3x - 4y = 25$ Ans: (3,-4)
    c) $xy = 8$, $y = x - 2$ Ans: (4,2);(-2,-4)
    d) $x^2+y^2=25$, $y = \frac{12}{x}$ Ans: (-3,-4), (3.4)
    e) $y = 4-x^2$, $x^2+y^2 = 16$ Ans: (0,4), (-4,0), (4,0)

Lesson: 42

AIM: How can we solve a quadratic-linear system algebraically?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Solve for y in the linear equation.
2. Substitute the expression for y in the quadratic equation.
3. Transform the quadratic equation into standard form.
4. Solve the resulting quadratic equation for x.
5. Substitute each value of x into the linear equation to find the corresponding value of y.
6. Check each pair of values in the original equations.

VOCABULARY: no new terms

MOTIVATION:

Solve graphically $y = x^2 + 2x + 1$
$y = x + 3$

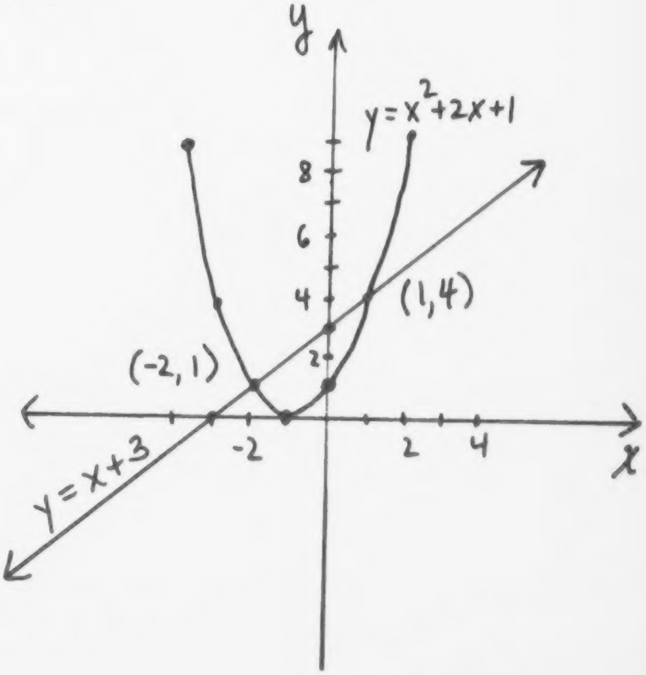
x	$x^2 + 2x + 1$	y
-4	$(-4)^2+2(-4)+1$	9
-3	$(-3)^2+2(-3)+1$	4
-2	$(-2)^2+2(-2)+1$	1
-1	$(-1)^2+2(-1)+1$	0
0	0 + 0 + 1	1
1	$(1)^2+2(1)+1$	4
2	$(2)^2+2(2)+1$	9

Axis of symmetry

$$x = \frac{-b}{2a} = \frac{-2}{2(1)} = -1$$

$y = x + 3$

x	x + 3	y
0	0 + 3	3
1	1 + 3	4
2	2 + 3	5



DEVELOPMENT:

Refer to motivation.

Let us check the solution by solving the system of equations algebraically. We know how to solve systems of linear equations algebraically. Let us now extend our knowledge to include quadratic-linear systems.

A system consisting of one linear and one quadratic equation can be solved by substitution. If the linear equation is solved for one variable in terms of the other, this result can be substituted in the quadratic. This will produce a quadratic in one variable.

Solve and check algebraically:

METHOD

1. Since y is the independent variable in the quadratic equation, substitute for y.
2. Write the quadratic equation in standard form. (Set equal to zero)
3. Solve the quadratic equation.
4. Substitute in the linear equation to find the corresponding value of y.
5. Write the solutions as ordered pairs of numbers..
6. Check in each of the original equations.

Solve and check:

$y = x^2 + 2x + 1$

$y = x + 3$

$x + 3 = x^2 + 2x + 1$ substitute

$0 = x^2 - x - 2$

$0 = (x - 2)(x + 1)$

$x - 2 = 0 \quad x + 1 = 0$

$x = 2 \quad x = -1$

$y = x + 3$ $y = 2 + 3$ $y = 5$	$y = x + 3$ $y = -1 + 3$ $y = 2$

$x = 2$ $y = 5$	$y = 5$ $y = 2$

Example 1:

Solve and check:

$x^2 + 2y = 5$

$y = x + 1$

$x^2 + 2(x + 1) = 5$

$x^2 + 2x + 2 = 5$

$x^2 + 2x - 3 = 0$

$(x - 1)(x + 3) = 0$

$x - 1 = 0 \quad x + 3 = 0$

$x = 1 \quad x = -3$

$y = x + 1$ $y = 1 + 1$ $y = 2$	$y = x + 1$ $y = -3 + 1$ $y = -2$

$x = 1$ $y = 2$	$x = -3$ $y = -2$

Example 2

Solve and check:

$y = x^2 - x$

$x + y = 4$

$x + x^2 - x = 4$

$x^2 = 4$

$x = 2$	$x = -2$
$x + y = 4$	$x + y = 4$
$2 + y = 4$	$-2 + y = 4$
$y = 2$	$y = 6$

$x = 2$ $y = 2$	$x = -2$ $y = 6$

Example 3

Solve and check:

$x^2 + y^2 = 25$

$x = 2y - 5$

$(2y - 5)^2 + y^2 = 25$

$4y^2 - 20y + 25 + y^2 = 25$

$5y^2 - 20y = 0$

$5y(y - 4) = 0$

$5y = 0$	$y - 4 = 0$
$y = 0$	$y = 4$
$x = 2y - 5$	$x = 2y - 5$
$x = 2(0) - 5$	$x = 2(4) - 5$
$x = -5$	$x = 8 - 5 = 3$

$y = 0$ $x = -5$	$y = 4$ $x = 3$

A straight line and the graph of a 2nd degree equation have two points, one point, or no point in common. If they have one point in common, the line is tangent to the curve; the corresponding system of equations has two identical roots. If they have no point in common, then the corresponding system of equations has no real roots in common, but the system does have imaginary roots.

Example 4: Solve

1) $x^2 - 2x - y + 1 = 0$ Ans: $\frac{7 - i\sqrt{59}}{6}$, $\frac{-29 - i\sqrt{59}}{18}$

2) $x - 3y - 6 = 0$ $\frac{7 + i\sqrt{59}}{6}$, $\frac{-29 + i\sqrt{59}}{18}$

The graph of equation 1) is a parabola and that of equation 2) is a straight line. Since the graphs have no points in common, the solution set consists of imaginary numbers. Solve $x - 3y - 6 = 0$ for x: $x = 3y + 6$. Then substitute in first equation, etc.

APPLICATIONS:

Solve each system and check.

1. $x^2 + y^2 = 25$
   $3x - 4y = 25$ Ans: (3,-4)
2. $x = y^2 - y + 1$ Ans: (1,1)
   $x + y = 2$ (3,-1)

3. $x^2 + y^2 = 9$
$y = -2x + 8$ Ans: $\emptyset$

SUMMARY:

1. What is the graphical interpretation of a linear-quadratic system which has:
   a) No real roots.
   b) One double root.
   c) Two distinct roots.
2. What advantage does an algebraic solution of a quadratic-linear system have compared to a graphical solution?

HOMEWORK:

Solve the system and check:

1. $2x - 7y = 9$ 2. $y = 5x^2 - 7x + 3$
$y = (x - 5)^2$ $4x - 3y - 1 = 0$
Ans: $(1,1); (\frac{2}{3}, \frac{5}{9})$

3. $3x + 2y = 6$
$9x^2 + 4y^2 = 36$ Ans: $(2,0); (0,3)$

4. $x^2 - y^2 = 9$ 5. $x^2 - y^2 = 9$
$y = -x + 9$ $y = -1 - x$
Ans: $(5,4)$ Ans: $(-5,4)$

6. Solve the following system of equations and check.
$y = 2x^2 + 3x - 1$
$y = 2x$

7. $xy = 12$ Ans: $(2,6)$ $(3,4)$
$4 + 2x = 10$

8. Find two numbers whose sum is 21, and the sum of their squares is 233.

9. Find the value of k so that the line is tangent to the curve.
a) $y = 5x - 21$ b) $x + k + y^2$
$y + 5x = x^2 + k$ $2y - x = 4$
Ans: $(k = 4)$ Ans: $(k = 3)$

Lesson: 43

AIM: How to solve geometric problems whose solutions lead to quadratic equations?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Identify and define the variables for geometric problems.
2. State and apply the appropriate geometric formula (area, similarity, perimeter, Pythagorean theorem) to set up an equation.
3. Solve the equation by factoring or by using the quadratic formula.
4. Check the solutions by substituting in the original statement of the problem.

VOCABULARY: no new terms

MOTIVATION:

The product of two consecutive positive integers is forty more than four times the smaller integer. What are the numbers?

DEVELOPMENT:

1. Review the technique of representing consecutive integers as well as consecutive even and consecutive odd integers. Let n = first positive integer, n+1 = second consecutive positive integer.

$$n(n+1) = 4n + 40$$
$$n^2 + n = 4n + 40$$
$$n^2 - 3n - 40 = 0$$
$$(n-8)(n+5) = 0$$

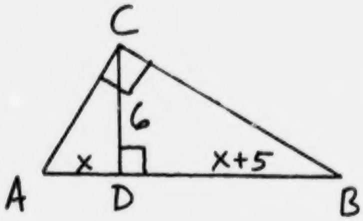
Then $n = 8$ or $n = -5$. The solution set for the equation is $\{8,-5\}$.

Examine the solution set for the quadratic equation. Since the integers in the original problem are specified to be positive, the root, -5, must be rejected. The only solution for n is 8. The two numbers are 8 and 9.

2. Steps to solve geometric problems using quadratic equations.
   a) Read the problem.
   b) Draw and use a diagram illustrating the given facts.
   c) Identify the appropirate geometric relationship, for example,the formulas for the area of a rectangle, a parallelogram, a triangle and a circle.
   d) Introduce appropriate variables to represent the quantities involved in the problem.
   e) Express the conditions of the problem as a quadratic equation (or system of equations).
   f) Solve the equation or (system of equations).
   g) Check the statement of the problem to see that the solution does actually satisfy the conditions of the problem.

Remember that a quadratic equation has two roots. It is entirely possible that one root, or possibly both, will be meaningless in the context of the problem.

3. In the accompanying diagram of right triangle ABC, altitude $\overline{CD}$ divides hypotenuse $\overline{AB}$ into segments of lengths x and (x+5). If CD = 6, find the length of hypotenuse $\overline{AB}$.



Solution: What relationship is needed to solve the above problem? In a right triangle, the length of the altitude drawn to the hypotenuse is the mean proportional between the lengths of the segments of the hypotenuse.

$$\frac{x}{6} = \frac{6}{x+5}$$

$$x(x+5) = 36$$

$$x^2 + 5x = 36$$

$$x^2 + 5x - 36 = 0$$

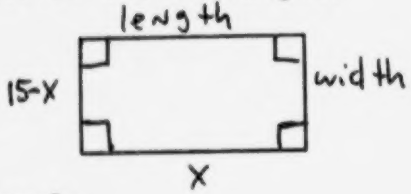
$$(x+9)(x-4) = 0$$

$$\begin{array}{c|c} x+9 = 0 & x-4 = 0 \\ x = -9 & x = 4 \\ \text{reject} & \text{Find AB} = 4+9=13 \end{array}$$

4. 30 feet of fencing will enclose a rectangular area of 54 square feet. Find the dimensions of the rectangle.

Draw a diagram.

Let x = number of inches in the length. Let (15-x) represent the number of inches in the width of the rectangle.



$A = lw$

$54 = x(15-x)$

$54 = 15x - x^2$

$x^2 - 15x + 54 = 0$

$(x-9)(x-6) = 0$

$x - 9 = 0$ or $x - 6 = 0$

$x = 9$ or $x = 6$

The solution set is $\{9,6\}$. If $x = 9$, then $15 - x = 6$. If $x = 6$, then $15 - x = 9$. The dimensions of the rectangle are 9 inches and 6 inches. Check the solution.

5. The hypotenuse of a right triangle is 10 inches long. The difference between the lengths of the other two legs is 2 inches. Find the lengths of the two legs of this triangle.

Let x = length of shorter leg = 6

x + 2 = length of other leg = 8

$$(x)^2+(x+2)^2 = 10^2$$

$$x^2 + x^2 + 4x + 4 = 100$$

$$2x^2 + 4x - 96 = 0$$

$$2(x^2+2x-48) = 0$$

$$(x+8)(x-6) = 0$$

$$\begin{array}{r|l} x + 8 = 0 & x - 6 = 0 \\ \text{reject } x = -8 & x = 6 \end{array}$$

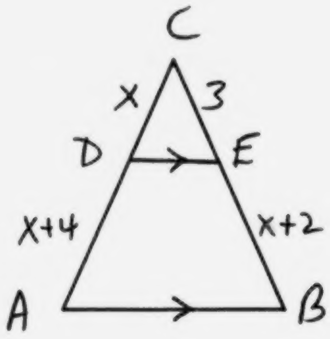
After solving the quadratic equation, examine the roots to determine whether any do NOT meet the conditions of the problem. Verify by checking the roots in the wording of the problem.

APPLICATIONS:

1. The hypotenuse of a right triangle measures 25. The sum of the lengths of the legs is 31. Find the length of the legs of the triangle. Ans: 24 and 7.

2. Find two consecutive odd integers whose product is 323. Ans: -19 and -17 or 19 and 17

3. $\triangle$ABC is given in which D is a point on $\overline{AC}$, E is a point on $\overline{BC}$, and $\overline{DE}$ $\overline{AB}$. If CE = 3, EB = x+2, CD = x and DA = x+4, find CD.



$$\frac{x}{x+4} = \frac{3}{x+2}$$

$$3(x+4) = x(x+2)$$

$$3x + 12 = x^2 + 2x$$

$$0 = x^2 - x - 12$$

$$\begin{array}{r|l} x - 4 = 0 & x + 3 = 0 \\ x = 4 & x = -3 \quad \text{reject} \end{array}$$

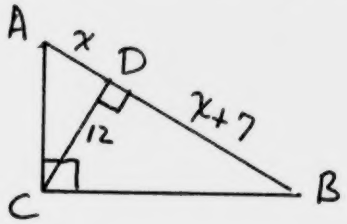
4. The perimeter of a rectangle is 20 and the length of a diagonal is 8.
   a) If x represents the width of the rectangle, represent the length in terms of x.

b) Write and solve an equation to find the dimensions of the rectangle.

HOMEWORK:

1. The sum of the squares of two consecutive odd integers is 202. Find these two integers.
2. The measures of the base and height of a parallelogram are in the ratio 2:3. If its area is 54 square inches, find the dimensions of the parallelogram. (Review the formula for area of a parallelogram.)
3. A supermarket, rectangular in shape and 200 feet by 300 feet, is to be built on a city block that contains 81,600 square feet. There will be a uniform strip around the building for parking. How wide is the strip?

   HINT: If the strip is 5 feet wide, the dimensions of the supermarket will be 200+25 by 300+25. Its area is (200+25)(300+25) square feet. But the area is given as 81,600 square feet.
4. A rectangle is equal in area to a triangle whose base measures 12 inches and its height measures 11 inches. If the base of the rectangle is 5 inches longer than its height, find the dimensions of the rectangle. (Review the formula for area of a triangle.)
5. In the accompanying diagram of right triangle ABC, $\overline{CD}$ is the altitude to hypotenuse $\overline{AB}$. If CD = 12, AD =x, and BD is 7 more than AD, find x.



$$\frac{x}{12} = \frac{12}{x+7}$$

$$x(x + 7) = (12)(12)$$

$$x^2 + 7x = 144$$

$$(x+16)(x-9) = 0$$

$$x + 16 = 0 \qquad x - 9 = 0$$

$$\text{reject } x = -16 \qquad x = 9$$

$$\text{Ans: } x = 9$$

6. The length of the diagonal of a rectangle is $2\sqrt{5}$ cm. If the area of the rectangle is 8 cm^2, find the dimensions of the rectangle. (Ans: 2 cm and 4 cm)

7. The product of two numbers is 153. The sum of their squares is 370. Find the numbers. (Ans: 9 and 17)

Lesson 44

AIM: How can we solve simple probability problems?

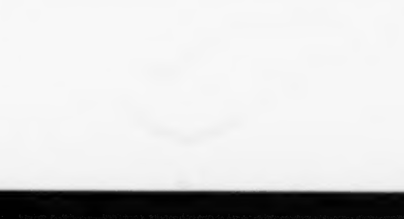
PERFORMANCE OBJECTIVES: The student will be able to...

1. Define a sample space.
2. Define an event.
3. State and apply the formula for the theoretical probability of an event.
4. State the range of possible values for the probability of an event.
5. Define uniform probability.
6. Apply the formula for finding the probability of two events, using the "or" relationship.
7. Use the concepts of union and intersection of sets to explain the "or" formula.

VOCABULARY: sample space, outcome, uniform probability, union, intersection, mutually exclusive, random

MOTIVATION:

If we toss a single six faced die, what is the probability of obtaining an even number? A number greater than 4? (2/6) List the sample space (1,2,3,4,5,6) and "favorable" outcomes of the first event (2,4,6) which we call event A. Define event B as the outcomes "number is greater than 4", i.e., (5,6). Therefore P(A) = 3/6, P(B) = 2/6. Demonstrate that this can be pictured in a Venn Diagram such as:



How do we denote the overlapping area? ($A \cap B$) The total area within the two circles? (AUB) What is P($A \cap B$)? (1/6). What is P(AUB)? (4/6).

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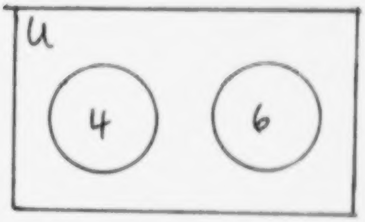
DEVELOPMENT:

Note that the probability of an event is the number of ways the event can happen divided by the number of outcomes in the sample space $P(E) = \frac{n(E)}{n(S)}$ or in this case $P(B) = \frac{n(B)}{n(S)}$. Ask, "What could be the lowest and highest value of this probability?" (0,1) Note that the larger the fraction between 0 and 1, the more likely the event is to occur. In general, $0 \leq P(A) \leq 1$. Mention complementary probability. Illustrate that $P(A') = 1-P(A)$. Examples such as why the probability of "head" on a coin is 1/2, i.e., head, tail are equally likely, or the probability of "4" on a single tossed die - 1,2,3,4,5,6 are equally likely, may be included to discuss the uniform probability (the expectation that all outcomes are equally likely) of an event. Discuss the term "outcomes" in these contexts. If we toss a die, what is the probability of a "six"? (1/6) ...a "four"? (1/6) $P(6) = 1/6$, $P(4) = 1/6$. Tossing a 6 or 4? Students will correctly state 2/6, e.g., $P(4 \text{ or } 6) = p(4) = 1/6 + 1/6 = 2/6 = 1/3$. Recall the motivation problem and apply the concepts of union and intersection to the above problem, initially using $P(4 \cup 6)$. Ask, "What is the probability of $P(4 \cap 6)$ in one toss?" Note $P(4 \cap 6) = 0$ since 4,6 are "mutually exclusive" events, that is, events which cannot happen at the same time. Ask, "How does this compare to sets A and B in the motivation?"



Poll the class on how many students are wearing anything blue. Express the probability that a student randomly selected from the class will be wearing anything blue as a ratio, i.e., 21/34 = P(blue). Next ask the class what is the probability of a "randomly" selected student in the class being male. For example, if there are 16 boys in a class of 34, then $P(M) = \frac{16}{34}$. Ask, "What is the sample space in this experiment? (The students in this class.) In the previous two examples, what events are we considering? (Wearing blue or being male.)

What do the 21 and 34 represent in the first example? What is the probability of selecting a student randomly and finding the student is either a male or someone wearing blue?

Note: $P(M \cup B) = P(M) + P(B) = \frac{16}{34} + \frac{21}{34} = \frac{37}{34} > 1$

which is impossible.

Ask, "How is it possible that there could be 37 people satisfying either requirement when there are only 34 in the class?" (Some were counted twice.) Ask the class for these students to raise their hands. If, for example, there are 10 in this category, then $P(M \cap B) = 10/34$. Note that the 10 people who are in both categories were counted twice and therefore 10/34 must be subtracted from 37/34, that is, $\frac{37}{34} - \frac{10}{34} = \frac{27}{34}$ is the answer to the question. This can be confirmed by polling the class. In general terms, students should then (re) discover $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Note in this example "male" and "wearing blue" are not mutually exclusive and therefore $P(M \cap B) \neq 0$.

APPLICATIONS:

1. Use the above formula to find, in the motivation problem $P(A \cup B)$. $(\frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6})$.
2. A card is drawn from a standard deck. Find the probability of drawing
   (a) heart or a queen (b) an ace or picture card
   (c) a jack, king, or spade
3. The integers from 5 to 12 inclusive are written on separate pieces of paper and placed in a hat. One piece of paper is selected at random and examined. Find the probability that the number is:
   a. Even or divisible by 3.
   b. Odd or greater than 10.
   c. Even or prime.
   d. The median or the range of the numbers in the hat.
   e. Divisible by 2, 3, or 4.

SUMMARY:

1. What is the probability of A or B?
2. What does mutually exclusive mean?
3. When is $P(A \cup B) = P(A) + P(B)$?
4. If A represents the event that "the sun will rise tomorrow," what is $P(A)$? $P(A')$?

HOMEWORK:

1. In a deck of 52 cards, find for one selection: P(Jack), $(\frac{4}{52})$, P(Red), $(\frac{26}{52})$, P(Red Jack), $(\frac{2}{52})$ P(Club), $(\frac{16}{52})$, P(Picture U Club), $(\frac{16}{52} + \frac{12}{52} - \frac{3}{52} = \frac{25}{52})$
2. If n is a positive integer less than 11, find, assuming uniform probability, $P(\frac{n}{2} = 4)$, $(\frac{1}{10})$, $P(\frac{n}{2} > 5)$, (0), $P(\frac{n}{3} < 1)$, $(\frac{2}{10})$
3. An urn contains 5 white marbles and 3 black. If one marble is chosen at random, find P(white), $(\frac{5}{8})$, P(black), $(\frac{3}{8})$, $P(\text{white} \cup \text{black})$, (1), $P(\text{white} \cap \text{black})$, (0).
4. The integers from 10 to 21 inclusive are written on separate pieces of paper and placed in a hat. One piece of paper is selected at random and examined. Find the probability that the number is
   a. divisible by 3
   b. greater than 14
   c. less than or equal to 17
   d. prime or odd
   e. divisible by 7 or even
   f. less than 21 and greater than 10
   g. a perfect square
   h. the average of the numbers or the median of the numbers in the hat.

5. Given four distinct lines, exactly two of which are parallel, which of the following could be the number of points where at least two of the lines intersect?

I. Three
II. Four
III. Five

(A) I only (B) III only (C) I and II only
(D) I and III only (E) I, II, and III

Lesson 45

AIM: How can we use the counting principle to solve probability problems?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Explain and apply the counting principle.
2. Define a compound event.
3. Use a tree diagram or sample space to solve probability problems.
4. Explain and apply the counting principle involving the probability of events occurring jointly.

VOCABULARY: counting principle, compound event, independent events

MOTIVATION:

Suppose there are four teachers of Course II and three teachers of Biology. Ask, "How many ways may a student have different teachers for these two subjects?" (12) Discuss that the answer (4·3) is an illustration of the "counting principle", and that the "event" of having a particular teacher in one subject is "independent" of having a particular teacher in the other subject. Independent events are those in which the outcome of any one event does not effect the outcome of the others. Define compound event as one which consists of two or more independent events.

DEVELOPMENT

Give a formal definition of the Counting Principle.

THE COUNTING PRINCIPLE

If E_1 and E_2 are independent events and E_1 can occur in M ways and E_2 can occur in N ways, then the number of ways E_1 can occur followed by E_2 is M x N.

Example 1

E_1 = flipping a coin and E_2 = selecting a marble from an urn that contains one yellow, one blue, and one green marble. E_1 and E_2 are independent events. if E_1 occurs followed by E_2, then the number of different outcomes of these two events is

2(H or T) x 3(Y,B, or G) = 6

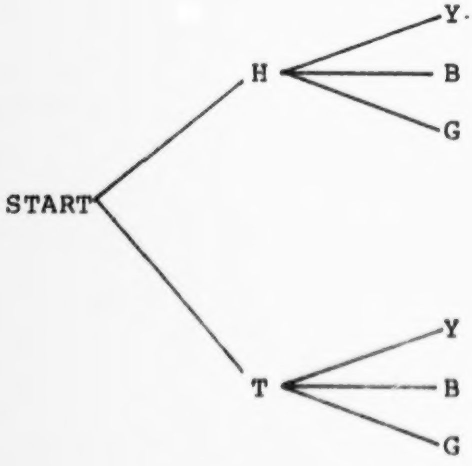
Example 2

E_1 = flipping a coin and E_2 rolling a dice. E_1 and E_2 are independent events and may occur jointly in

2(H or T) x 6 (1,2,3,4,5, or 6) = 12 ways.

In each case the class should be asked to list the elements of the sample space.

Refer back to example 1 and ask, (a) "what is the probability of tossing a tail on a coin and selecting a green marble from the urn?" Have the class draw a tree diagram:



P(J and G) = $\frac{1}{6}$

Explain that the denominator is the numnber of elements in the sample space which is found by applying the counting principle to E_1 and E_2 (2·3=6).

The numerator is the counting principle applied to the compound event of getting a tail followed by selecting a green marble ($1\cdot1=1$).

Next, refer back to example 2 and ask, "what is the probability of tossing a head on a coin and rolling a 4 on a die?" Ask students to apply the counting principle,

$$P(H \cap 4) = \frac{1\cdot1}{2\cdot6} = \frac{1}{12}$$

Also point out that,

$$P(H \cap 4) = \frac{1\cdot1}{2\cdot6} = \frac{1}{2} \cdot \frac{1}{6} = P(H) \cdot P(4)$$

In general, If A and B are independent events, then

$$P(A \cap B) = P(A) \cdot P(B)$$

APPLICATIONS:

1. Find the probability of rolling a three on a single die and then picking a red card from a deck. $(\frac{1}{6} \cdot \frac{1}{2})$
2. Find the probability of tossing three successive heads on a coin $(\frac{1}{2})^3$.
3. A single die is rolled and then a card is selected from a standard deck. Find the probability of
   a) rolling a six and selecting a picture card. $(\frac{1}{6} \cdot \frac{12}{52})$
   b) rolling an even number and selecting an ace $(\frac{3}{6} \cdot \frac{4}{52})$
   c) rolling a number less than 5 and selecting a heart whose face value is between 3 and 10 $(\frac{4}{6} \cdot \frac{24}{52})$

SUMMARY:

1. If one event can occur m ways, another n ways, and a third event in p ways, how many ways can the three events occur?
2. What is a compound event?
3. Ask, "To find the probability of having an automobile accident while it is icy, can we use $P(\text{Accident} \cap \text{Icy}) = P(\text{Accident}) \cdot P(\text{Icy})$?" Why? (They are not independent since icy roads increase P(Accident).

HOMEWORK:

1. One must choose between three subway lines, followed by 4 bus routes, to reach a certain destination. How many different ways can this place be reached? (12)
2. An urn contains 5 red marbles, 2 blue and 3 yellow. A marble is selected, color noted and then replaced. Another is selected, etc.
   Find $P(\text{red} \cap \text{blue})$ (in order shown) $(\frac{5}{10} \cdot \frac{2}{10} = \frac{1}{10})$
   $P(\text{red} \cap \text{yellow})$ (in order shown) $(\frac{5}{10} \cdot \frac{3}{10} = \frac{3}{20})$
   $P(\text{yellow} \cap \text{yellow})$ (in order shown) $(\frac{3}{10} \cdot \frac{3}{10} = \frac{9}{100})$
   $P(\text{blue} \cap \text{yellow} \cap \text{red})$ (in order shown) $(\frac{2}{10} \cdot \frac{3}{10} \cdot \frac{5}{10} = \frac{3}{100})$
   (Discuss if order is not important.)
3. If x(2,4,5) and x is randomly selected twice from this set, find
   a. $P\ (2x=8 \cap x+1=5)$ Ans. $(\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9})$
   b. $P\ (3x=6 \cap x+6=20)$ Ans. $(\frac{1}{3} \cdot 0 = 0)$
   c. $P\ (x<6 \cap x \text{ is even})$ Ans. $(1 \cdot \frac{2}{3} = \frac{2}{3})$

Lesson 46

AIM: How can we count the possible arrangements of a set of objects in a particular order?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define a permutation.
2. Use the counting principle to derive the permutation formula for n objects taken r at a time.
3. Apply the permutation formula to count the number of possible numerals that can be formed from digits and the number of possible words that can be formed from letters.

VOCABULARY: permutation, factorial

MOTIVATION:

Ask, "How many ways may the letters a,b,c,d be arranged?" After having students give examples of possible arrangements, ask what word describes such an arrangement. Note the definition of "permutation" as an arrangement in which order counts. Ask what principle gives the total number of such arrangements (the fundamental counting principle). Therefore, since we have 4 choices for the first letter, 3 for the second, etc., the total is (4)(3)(2)(1) = 24 = 4! (review the meaning of n! as n(n-1)(n-2)...1)

Note: ${}_4P_4 = 4!$ means the number of permutations of 4 objects when all are used

${}_nP_n = n!$ means the number of permutations of n objects when all are used

DEVELOPMENT:

Suppose only two letters are to be used. Ask, "Would there be more or fewer such permutations? Why?" Demonstrate that all the permutations (using four letters) starting with bd for example (bdac and bdca) would now be the same (bd) when only two letters are used. Therefore, ${}_4P_4 = 2!({}_4P_2)$ where ${}_4P_2$ denotes the number of permutations of 4 objects when only 2 are used. Since ${}_4P_4 = 4!$, ${}_4P_2 = \frac{{}_4P_4}{2!} = \frac{4!}{2!}$ Ask the class to compare how many arrangements of five objects taken two at a time there are with the number when all five objects are used. For instance, bdace is one of six or 3! permutations starting with bd which would all be the same when the last three letters are omitted. Therefore, ${}_5P_5 = 3! \cdot {}_5P_2$. In general, ${}_nP_n = (n-r)! \cdot {}_nP_r$, or ${}_nP_r = \frac{{}_nP_n}{(n-r)!} = \frac{n!}{(n-r)!}$ where ${}_nP_r$ denotes the number of permutations or r objects, of which r are used. Also note that

$${}_nP_r = \underbrace{n(n-1)(n-2)\ \ldots\ (n-r+1)}_{r\text{ factors}}$$

For example, ${}_8P_3 = 8\cdot7\cdot6 = 336$

APPLICATIONS:

1. How many ways may four people stand on line?" (4!)
2. How many different license plate numbers using the digits 0 through 9 may be formed for license plate numbers of 6 digits (each digit may be used only once)? $(\frac{10!}{3!})$
3. How many three letter "words" can be formed from the letters BASIC (use no letter more than once)?" $(\frac{5!}{2!})$
4. There are 9 players on a baseball team. (a) How many different batting orders are possible if the pitcher always bats last? (8!). (b) In how many different ways can these players seat themselves in a row that contains 12 chairs? $(\frac{12!}{3!})$

SUMMARY:

1. What is a permutation?
2. How do we compute the number of permutations of n objects if all are used? If only r are used?
3. Solve for x: ${}_xP_2 = 7x+9$

HOMEWORK:

1. Calculate ${}_5P_5$, ${}_5P_4$, ${}_5P_3$, ${}_5P_2$, ${}_5P_1$, ${}_5P_0$ (use $0!=1$)
   (120) (120) (60) (20) (5) (1)

2. How many ways may five students be seated in a row of four seats? (${}_5P_4 = 120$) ...of 7 seats? (7P5 = 2520)

3. A license plate consists of 3 letters followed by 3 digits, excluding zero. How many different plates may be issued? ($(26)^3 \cdot (9)^3$)

4. How many rows would a truth table have if there were three propositions: p,q,r? ($2^3=8$)

5. How many telephone numbers start with
   (a) 718-596________? (10^4)
   (b) 212-___________? (10^7)

6. How many different words of at most 3 letters can be made from the letters in PARIS? $({}_5P_1+{}_5P_2+{}_5P_3 = \frac{5!}{4!} + \frac{5!}{3!} + \frac{5!}{2!})$

   At least 3 letters from PARIS? $({}_5P_3+{}_5P_4+{}_5P_5 = \frac{5!}{2!} + \frac{5!}{1!} + \frac{5!}{0!})$

7. From the set of digits 1,2,3,4,5 , how many 3-digit numbers can be formed if no digit is repeated and the number is
   a. between 100 and 600 (${}_5P_3$)
   b. greater than 200 (${}_4P_3$)
   c. less than 500 ($4 \cdot {}_4P_2$)
   d. odd ($3 \cdot {}_4P_2$)

8. Solve for y: ${}_yP_3 = y^3-18y-7$

Lesson 47

AIM: How can we count the possible arrangements of a set of objects, which are not all different, in a particular order?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Derive a formula for the number of permutations of n objects with r of them identical.
2. Apply the above formula to numeral and word problems.
3. Derive a formula for the number of permutations of n objects, with r_1 identical, r_2 identical and so on.
4. Apply the above formula to numeral and word problems.

VOCABULARY: permutations with repetitions

MOTIVATION:

How many different "words" may be formed by rearranging the letters of BAT? (3! = 6). The word BOB? Students should see that there are fewer of the latter, namely 3. Asking why they are fewer should lead to the discovery that this is due to the repetition of B; that is, two of the letters are identical. If the two B's were different (they may be labeled B_1, B_2), then OBB, for example, would produce 2! permutations: OB_1B_2 and OB_2B_1. Therefore there are 2! times as many permutations of 3 different letters as there are of 3 letters of which two are identical. That is, the number of different arrangements of the letters of the word Bob can be calculated by finding

$$\frac{(\text{number of letters})!}{(\text{number of Bs})!} = \frac{3!}{2!} = 3$$

DEVELOPMENT:

Now suppose we wish to find the number of permutations of the letters aaabb. Ask, "How does this differ from the previous problem? (Two letters are repeated.) If the a's were different, how many permutations would they produce? (3!) The b's?" (2!)

Therefore the number of permutations of five different objects = 3! 2! (The number of permutations of five objects of which three are alike and two are alike.) Therefore, the latter =

$$\frac{\text{\# of permutations of 5 different objects}}{(\text{\# of as})!\cdot(\text{\# of bs})!} = \frac{5!}{3!2!}$$

Now suppose we wish to find the number of permutations of n objects of which r_1 are alike, r_2 alike, etc....

This would be $\frac{n!}{r_1!r_2!\ldots}$

The permutation of a large word like MISSISSIPPPI may be included at this time as an illustration that factorials may be reduced:

$$\left(\frac{11!}{4!2!2!} = \frac{11x\overset{5}{\cancel{10}}x9x\overset{4}{\cancel{8}}x7x6x5}{\underset{1}{\cancel{2}x\cancel{2}}}\right)$$

APPLICATIONS:

1. In how many orders may two identical red flags, 5 identical white flags and three identical black flags be arranged in a line?" $\left(\frac{10!}{4!2!2!}\right)$
2. If x^2y^3 is written in expanded form, in how many ways may the letters be arranged? $\left(\frac{5!}{3!2!}\right)$
3. How many ways may the digits of the number 122333 be arranged? $\left(\frac{6!}{3!2!}\right)$
4. A bookshelf contains 3 copies of a certain algebra text, 4 copies of a certain geometry text, and 2 copies of a certain trigonometry text.
   (a) In how many different ways can these books be arranged on the shelf? $\left(\frac{9!}{3!4!2!}\right)$
   (b) If the trigonometry texts must be in front of the other texts, in how many different ways can these books be arranged on the shelf? $\left(\frac{7!}{3!4!}\right)$

5. Four of the same geometry books, three of the same calculus books, and five different computer books are to be placed on the shelf. In how many different ways can these books be arranged on the shelf if all the geometry books must be together? $(\frac{9!}{3!})$.

SUMMARY:

1. Give an example of a permutation with repetitions.
2. How may we compute the number of such permutations?
3. The binary representation of the number 25 is 11001. In how many different ways can a student have written this number incorrectly?

HOMEWORK:

1. Compute:

   a) $\frac{5!}{3!}$ (20)

   b) $\frac{8!}{5!}$ (336)

   c) $\frac{4!}{0!}$ (24)

2. Find the number of permutations of the letters of the word isosceles. $(\frac{9!}{3!2!} = 30{,}240)$
3. Find the number of permutations of 7 objects, 4 alike and 3 alike. $(\frac{7!}{4!3!} = 35)$
4. How many ways may the letters in the word DIFFICULT be arranged (express in factorial form). $(\frac{9!}{2!2!})$
5. In how many different ways may the letters XXXX be arranged? (1) XXXY? (4) XXYY? (6) XYYY? (4) YYYY? (1) Note: the answers here represent the coefficients in the binomial expansion of $(X+Y)^4$, to be studied in Course III.

Lesson 48

AIM: How can we use permutations to solve probability problems?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Explain how permutations can be used to count the number of outcomes of an event and the number of elements in a sample space.
2. Apply the permutation formulas to solve numeral and word probability problems.

VOCABULARY: no new terms

MOTIVATION:

A variable, x or y, is randomly selected in order from each parentheses: (x+y)(x+y)(x+y) and written as a product. For instance, xyx and xyy are two possible products. How many such orders are possible? Recall the fundamental counting principle to obtain $2\text{x}2\text{x}2 = 2^3$. Write all such orders.
What is the probability that the product is x^3? ($\frac{1}{8}$)
What is the probability that the product is x^2y? ($\frac{3}{8}$)

DEVELOPMENT:

The last question should lead to consideration of terms like xyx, xxy, yxx. Note that these are permutations which contain repititions. Ask, "What formula would count these for us?" ($\frac{n!}{r_1!} = \frac{3!}{2!} = 3$)
Therefore the probability that the product is equal to x^2y is $\frac{3}{8}$. In this example the number of elements in the sample space were given to us by formula for permutations.
Further examples involving the use of permutations in probability problems may be included such as the following:
The letters aabcd are rearranged at random. Ask, "What is the probability that the two a's will be in the last two positions?"

Note here the number of permutations is $\frac{5!}{2!} = 60$. The number of "words" ending with aa are 3x2x1 = 6. Therefore the answer is $\frac{6}{60} = \frac{1}{10}$.

APPLICATIONS:

1. A three digit numeral is formed by selecting from the digits 2,3,4,7 with no repetitions. Ask, "What is the probability the number formed is greater than 300?" Students should see that the answer, $\frac{3}{4}$, can be obtained immediately by using the digits 3,4,7 as the hundreds digit. The answer can also be obtained by considering the total number of such 3-digit numbers as 4x3x2 = 24. The numbers greater than 300 must begin with either 3, 4 or 7 followed by any one of three other digits, etc. = 3x3x2 = 18. Therefore, $\frac{18}{24}$ is the required answer.
2. Two cards are drawn from a standard deck of playing cards without replacement.
   - (a) Express the number of elements in the sample space using the notation ${}_nP_r$. $({}_{52}P_2)$
   - (b) What is the probability that the two cards selected are aces? $\left(\frac{{}_4P_2}{{}_{52}P_2} = \frac{1}{221}\right)$
   - (c) What is the probability that the two cards are the ace and king of hearts in that order?

$$\left(\frac{{}_1P_1 \cdot {}_1P_1}{{}_{52}P_2} = \frac{1}{52 \cdot 51}\right)$$

3. An urn contains 7 blue marbles, 3 yellow marbles, and 5 green marbles. Three marbles are selected from the urn without replacement. Find the probability that the marbles selected are 2 blue marbles and 1 green marble in that order. $\left(\frac{{}_7P_2 \cdot {}_5P_1}{{}_{15}P_3}\right)$
4. If there are two brothers in a group of 5 boys and these boys are lined up at random, what is the probability that the brothers will be next to each other? $\left(\frac{4!2!}{{}_5P_5}\right)$

SUMMARY:

How can the probability of an event be computed? (We must take the ratio of the number of ways an event may occur to the number of total possible outcomes. To find these, we need the fundamental counting principle and the formulas for permutations.)

HOMEWORK:

1. What is the total number of ways a coin that is tossed 5 times may fall? (25)
2. What is the probability of tossing a coin 5 times and obtaining HTHTT in exactly that order? $(\frac{1}{25})$
3. a) In how many orders may a coin show 3 tails and 2 heads after being tossed 5 times? $(\frac{5!}{3!2!})$
   b) Suppose we know there are three tails and two heads, what is the probability that the order was exactly HTHTT? $1 \div (\frac{5!}{3!2!})$
4. Three cards are drawn from a standard deck of playing cards without replacement. Find the probability that the cards are one picture card and two aces in that order.

   $$\left(\frac{{}_{16}P_{1}\cdot{}_{4}P_{3}}{{}_{52}P_{3}}\right)$$
5. The letters of the alphabet are rearranged randomly. What is the probability the first five letters are vowels? $(\frac{5!\cdot 21!}{26!})$

Lesson 49

AIM: How can we use permutations to solve linear and circular seating probability problems?

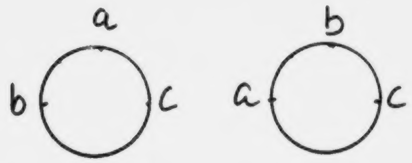
PERFORMANCE OBJECTIVES: The student will be able to...

1. Apply the permutation formula to count the number of arrangements of n people in a line.
2. Apply the permutation formula to determine the probability that r people out of n will be seated together in a line. (optional)
3. Count the number of arrangements of n people in a circle. (optional)
4. Determine the probability that r people out of n will be seated together in a circle. (optional)

VOCABULARY: no new terms

MOTIVATION:

Ask, "How many ways may three people wait in a line?" By denoting the people as a,b,c, students should list 3! = 6 permutations. Now suppose the people are seated around a circular table. Sketch all possible arrangements. Note that there are only 2!



This is because as far as person "a" is concerned, either b is on his right or c is on his right. There are no other possibilities.

DEVELOPMENT:

Extend the above to four people and have the students draw all possible circular permutations. Note the result is 3! = 6. Suppose there are n people. By considering person "a" in a fixed position, then the remaining n-1 people may be permutated (n-1)! ways (cut the circle and "bend" to form a

straight line to demonstrate this).
Thus n people may be permutated $n!$ in a straight line and $(n-1)!$ in a circle. For 5 people, the results would be $5!$ and $4!$, respectively.

Now consider if 2 of the above must be placed together. Begin with a straight line of five "people" (a,b,c,d,e) of which d and e must be together. They can be considered as a "unit" which can be moved to 4 positions; e.g., (de,-,-,-) or (-,de,-,-) or (-,-,de,-) or (-,-,-,de), that is $5-2+1 = 4$ ways. Notice there are $5-2 = 3$ blank positions for the other people plus one position for the "unit" of the two which must be together. However, the 2 people can themselves be arranged in $2!$ ways for each placement while the remaining 3 people can be arranged in $3!$ ways. Thus the product $(5-2+1)\ 4\cdot 2!\cdot 3! = 4!\cdot 2! = 48 = (n-r+1)!r!$ represents the total number of permutations. The probability that d,e will be together is $\frac{48}{5!} = \frac{48}{120} = \frac{2}{5}$.

In general, if we have n people in a straight line, r of which must stay together. The r people may be placed in $n-r+1$ positions. Within each position, they may be placed in $r!$ orders, while the remaining $n-r$ people may be placed in $(n-r)!$ orders. Thus the product $(n-r+1)(r!)(n-r)! = (n-r+1)!r!$ represents the total number of permutations. Now consider, if there are 5 people, 2 of which must be placed together around a circle. There are in effect $5-2$ others plus the "block" of the 2 who must be placed together. If we consider the "block" as one person, then there are $5-2+1 = 4$ people who may be permuted in $3!$ ways. To find the total permutations we must multiply by $2!$ (for the permutations within the "block") to obtain $3!2! = 12$. Thus the probability of 2 people being together in a circle of five people is $\frac{12}{4!} = \frac{12}{24} = \frac{1}{2}$.

In general, if r people must be together in a circle of n people, then $(n-r+1-1)!\cdot r! = (n-r)!r!$ gives us the number of ways they may be permutated. The probability of these r people being together around a circle is $\frac{(n-r)!r!}{(n-1)!}$.

APPLICATIONS:

Find the number of arrangements of 6 people if:

1. the people are lined up in a row. (6!)
2. the people are seated around a circular table (5!)
3. the people are seated around a circular table so that a particular person occupies a particular chair. (4!)

   If the group of 6 people consist of 3 men and 3 women, find the number of arrangements if:

4. All the men are to be next to each other when lined up.(4! · 3!)
5. All the women are to be seated next to each other when seated around a circular table (3! · 3!)
6. The men and women are to be seated alternately around a circular table. (2! · 3!)
7. In how many ways can 5 charms be placed on a bracelet which has no clasp? ($\frac{5!}{2}$)

SUMMARY:

What is:

1. the number of permutations of n objects in a line? (n!)
2. the number of permutations of n objects in a line if r must be together? ((n-r + 1)!r!)
3. the number of permutations of n objects in a circle? ((n-1)!)
4. the number of permutations of n objects in a circle if r must be together? ((n-r)!r!)

HOMEWORK:

(Answers may be left in factorial form.)

1. Find the number of permutations of 7 objects in a line. (7!)
2. Find the number of permutations of 7 objects in a line if 3 must be together. (5!3!)
3. Find the number of permutations of 7 objects in a circle. (6!)
4. Find the number of permutations of 7 objects in a circle if 3 must be together. (4!3!)
5. Three men and three women are to be seated. Find the numbers of possible seating arrangements if the men are to sit in three adjacent chairs and the women are to sit in three adjacent chairs (a) in a line (b) in a circle.

Lesson 50

AIM: How can we count the possible arrangements of a set of objects where order is not important?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define a combination.
2. Derive a formula for the combination of n objects taken r at a time.
3. Relate the combination formula to the permutation formula.
4. Identify the notation $\binom{n}{r}$ or ${}_nC_r$.
5. Explain the circumstances under which a permutation should be used or under which a combination should be used.

VOCABULARY: combination

MOTIVATION:

There are three seats available on a plane and five people seeking to take the flight. Ask, "How many ways may three people be selected who will board the plane?" Emphasize that the order of the selection is not important - what matters is only which of the five are selected.

DEVELOPMENT:

Define a combination as a selection where order does not matter, only the identity of the elements in the choice, or set, is significant. Denote the five people as (a,b,c,d,e) and have the class list possible combinations. Note there are only 10. Ask the class how many permutations there would be ${}_5P_3 = \frac{5!}{2!} = 60$. Note there are 6 times as many permutations because each selection can be permuted 3!

ways. Introduce $_5C_3$ as notation for the number of combinations of 5 objects taken 3 at a time. In general, $\binom{n}{r}$ or $_nC_r$ denotes the number of combinations of n objects taken r at a time.

Note: $_5P_3 = 3!\,_5C_3$ or $_5C_3 = \frac{_5P_3}{3!} = \frac{5!}{3!2!} = 10$

In general, if r objects are selected from n, then nPr = $r!\,_nC_r$ or $_nC_r = \frac{_nP_r}{r!} = \frac{n!}{r!(n-r)!}$

Change the airline problem so that now there are only two seats available for five people. Ask, "How many ways may two be selected?" ($_5C_2$) Compute this and ask why this is also ten. Selecting two from five is equivalent to selecting the three that are not included, i.e., for each different combination of two there is a different combination of three. In general, $_nC_r = {_nC_n}-r$. Students should be asked to verify this from the formula. Ask, "Why is $_nC_n = 1$?"

APPLICATIONS:

1. Four students in a class of 34 are to receive extra credit for their excellent homework. Ask, "How many ways could this selection be made?" ($_{34}C_4 = \frac{34!}{4!30!}$) Thirty students are not to receive extra credit. Ask, "How many ways may this selection be made?" ($_{34}C_{30} = \frac{34!}{30!4!}$)
2. In a poetry class of 20 students, 6 recite each day. In how many ways can these be selected for one day. ($_{20}C_6$)
3. Find the value of n such that $_9P_n = 6(_9C_n)$ (n=3)
4. Given 9 points in a plane, no 3 of them are collinear. How many triangles can be formed with these points as vertices? ($_9C_3$)
5. A series of committees are to be selected from a group of 9 people consisting of 4 women and 5 men. In how many ways can a committee of 3 be selected so that:
    a. the committee includes a particular person?
    b. the committee excludes a particular person?
    c. the committee includes all women?
    d. the committee includes the same number of men as women?

SUMMARY:

1. What is the difference between a permutation and a combination?
2. Define combination and give the formula.

HOMEWORK:

1. Compute ${}_5C_0$ (1), ${}_5C_1$ (5), ${}_5C_2$ (10), ${}_5C_3$ (10), ${}_5C_4$ (5), ${}_5C_5$ (1)
2. Replace the ? with the correct integer.

   ${}_5C_0 = {}_5C_?$ (? = 5)

   ${}_5C_1 = {}_5C_?$ (? = 4)

   ${}_5C_? = {}_5C_3$ (? = 2)
3. Subsets are formed from the elements of the set A,B,C,D,E . To form each subset, a decision is made whether to include or exclude each element. How many different subsets may be formed? ($2^5 = 32$) Compare this to the sum of your answers to question #1.
4. An urn contains 5 blue marbles and 9 red marbles. In how many ways may 6 marbles be drawn so that:
   a. there are an equal number of blue and red marbles selected?
   b. there are at least 3 blue marbles selected?
   c. there are at most 1 red marble selected?

Lesson 51

AIM: How can we apply combinations to count possibilities?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Apply the combination formula to count the number of possible committees, subsets and card hands.
2. Apply the combination formula to determine the number of different license plates that can be formed using letters and digits.

VOCABULARY: no new terms

MOTIVATION:

What is the total number of subsets possible of a set of 5 elements? From the previous homework, students should have discovered that ${}_5C_0 + {}_5C_1 + {}_5C_2 + {}_5C_3 + {}_5C_4 + {}_5C_5 = 2^5 = 32$. Ask, "In general, how many subsets are possible if the set contains n elements?" (2^n) This should be verified from the fundamental counting principle.

DEVELOPMENT:

1. Ask, "How many different committees of 4 people can be made from a group of 7 students?" Note that only who is included is important - not the order of selection. Ask, "Is this a permutation or combination problem? Why?"
2. Ask, "How many different hands of five cards is it possible to hold?" (${}_{52}C_5$) Again this is a combination problem.

3. Ask, "How many different license plates consisting of three different letters followed by three different numerals is it possible to have?" Here, we may first decide which letters and numbers are to be used (this is a combination ${}_{2}C_{3} \cdot {}_{10}C_{3}$) and then consider the permutations, i.e., multiply $3! \cdot 3!$.
Note that the last problem can also be solved by only permutations - ${}_{26}P_{3} \cdot {}_{10}P_{3}$.

APPLICATIONS:

1. A school offers 15 Course II Math sections, 16 Biology sections and 30 English sections. If a student must take only one class of Course II, one of Biology and two classes of English, in how many ways may this be done? $({}_{15}C_{1} \cdot {}_{16}C_{1} \cdot {}_{30}C_{2})$
2. How many committees of 4 people can be formed from 10 people if Mr. Jones, one of the ten, must be on the committee? $({}_{9}C_{3})$
3. A series of committees are to be selected from a group of 9 people consisting of 4 women and 5 men. In how many ways can a committee of 3 be selected so that:
   a) The committee includes exactly one woman?
   b) The committee includes at least one man?
   c) The committee consists either of all men or all women?
4. An equal number of men and women signed up for a mixed doubles tennis tournament. If 100 different matches could be arranged, how many men and women signed up for the tournament? (5 men and 5 women)

SUMMARY:

To find the number of committees, subsets and card hands we use combinations. In some problems, why must we find what the number of combinations are and then consider permutations of these combinations?

HOMEWORK:

1. How many different 6-member committees can be chosen if 9 people are available for the committee? ($_9C_6$)

2. A committee has 2 male members and 3 female members.
   a) How many 3 person subcommittees can be formed having at least 2 females? ($_2C_1 \cdot {_3C_2} + {_3C_0} \cdot {_3C_3}$)
   b) How many 3 person subcommittees can be formed having exactly 1 male member? ($_2C_1 \cdot {_2C_2}$)
   c) Three cards are drawn from a deck of cards. Find the number of ways that this can be done. ($_{52}C_3$)
   d) Two picture cards are selected from the set of only Jacks, Kings and Queens. How many ways may this be done? ($_{12}C_2$)
   e) If everyone in a class of 34 shakes hands with everyone else, how many handshakes take place? ($_{34}C_2$)

Lesson 52

AIM: How can we use combinations to solve counting problems in geometry?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Apply the combination formula to count the possible number of line segments collecting points in a plane.
2. Determine the number of diagonals for various polygons.
3. Find the number of points of intersection of lines, no two of which are parallel and no three of which are concurrent.
4. Determine the number of different triangles that can be drawn using sets of points in a plane.

VOCABULARY: concurrent, collinear

MOTIVATION:

Draw on the board a set of points, no three of which are collinear. Ask how many straight line segments could be drawn connecting the points. The relation to the previous homework problems should become apparent, i.e., that this is a combination problem. If there are 34 points (students) then there are ${}_{34}C_2$ line segments (handshakes). Ask, "For n points, how many different line segments could be drawn? (${}_nC_2$) Why is this a combination rather than a permutation?" (Order is not important - the line segment joining point A to point B is the same as joining B to A.)

DEVELOPMENT:

1. Draw a circle, with points located on the circumference. Ask how many chords could be drawn between these points.

Again, for n points, ${}_nC_2$ should be obtained. Ask, "What other geometric figures would lead to the same result?" Any regular polygon will suffice if the sides themselves are included. Show this with polygons of various numbers of sides.

2. Have the class now consider a set of non parallel straight lines, no three of which are concurrent (intersecting at the same point). Ask, "What similar question could we answer here?" The class should deduce that by using combinations the number of points of intersections of the lines can be obtained. Again, ${}_nC_2$ will be the result. Discuss the necessity of the lines being nonparallel as well as no three being concurrent .

3. Another example of using combinations will be determining the number of different triangles that can be drawn using sets of points in the plane. Here, for n points, ${}_nC_3$ should be deduced as the number of possible triangles. Ask, "For this formula to work, why is it necessary that no three of the points be collinear?" (Those points would not form a triangle.)

APPLICATIONS:

1. Draw a circle with six points on the circle designated. Ask, "How many chords can be drawn connecting these points. (${}_6C_2$) How many triangles? (${}_6C_3$) Quadrilaterals? (${}_6C_4$) Pentagons? (${}_6C_5$) Hexagons?" (${}_6C_6 = 1$)
2. An octagon is a polygon having 8 sides.
   (a) How many different diagonals can be drawn?
   (b) How many triangles can be drawn if each vertice of the triangle is also a vertice of the octagon?
3. Show that for a n-sided polygon (n>3), the number of diagonals that can be drawn is given by the formula $\frac{n(n-3)}{2}$.

$$({}_nC_2 - n = \frac{n(n-3)}{2}.)$$

SUMMARY:

Combinations can be used to solve geometric problems involving the maximum possible line segments, triangles, quadrilaterals, etc. connecting sets of points, or points of intersections of line segments.

HOMEWORK:

1. Find the number of chords possible that connect the following number of points on a circle.
   - a) 2 (1)
   - b) 3 (3)
   - c) 4 (6)
   - d) 5 (10)
2. Find the number of intersection points for the following number of non-parallel lines, no three of which are concurrent.
   - a) 2 (1)
   - b) 3 (3)
   - c) 6 (15)
   - d) 7 (21)
3. Find the number of different triangles possible formed by joining the number points, no three of which are collinear.
   - a) 3 (1)
   - b) 4 (3)
   - c) 5 (10)
   - d) 6 (20)
4. Find the number of diagonals that can be drawing for each of the following polygons:
   - a) hexagon
   - b) decagon
   - c) pentagon

Lesson 53

AIM: How can we use combinations to solve probability problems?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Explain under what circumstances combinations should be used to count the number of elements in an event and in a sample space.
2. Apply the combination formula to solve committee, subset and matching of elements from urn probability problems.

VOCABULARY: no new terms

MOTIVATION:

Juan, Paul, Jane and Sarah have volunteered to work on a committee that will require only two members. The two members are selected at random. Ask, "How many ways may this be done? ($_4C_2 = 6$) What is the probability that Juan and Sarah will be chosen?" ($\frac{1}{6}$)

In response to the first question, students should be asked to write the sample space corresponding to the event. It should be apparent that order does not matter and hence this is a combination problem.

For the second question, students should be asked the definition of probability; the formula $P(E)=\frac{n(E)}{n(S)}$ should be recalled.

DEVELOPMENT:

1. Three cards are drawn from a deck of 52 cards without replacement. Ask, "What is the probability that all are aces?" As in the previous problem, n(S) should be obtained. In this case, this is $_{52}C_3$ as again the order of drawing the cards is not important. The concept that there are $_4C_3$ combinations of ways of drawing the three aces is not as apparent.

It should be stressed that in order to obtain n(E) we are concerned only with the four ace subset of the deck. The three aces must be seleted from this group of 4. From the definition of probability, $P(E) = \frac{{}_4C_3}{{}_{52}C_3} = \frac{4}{221,000}$.

2. Next consider an urn that has 5 black and 3 white marbles. Four marbles are chosen at random. Ask, "What is the probability that two are white and two are black?" As in the card problem, order is unimportant and therefore $n(S) = {}_8C_4$. Ask, "How many ways may the two white marbles be drawn? (${}_3C_2$) The black marbles?" (${}_5C_2$) Recall here the fundamental counting principle to obtain $n(E) = {}_3C_2 \cdot {}_5C_2$. Then $P(E) = \frac{{}_3C_2 \cdot {}_5C_2}{{}_8C_4}$. The class should compute the result.

APPLICATIONS:

1. For the above problem, compute the probability that at least three are black if five marbles are chosen at random. Ask, "How many ways could this happen?" (3 black, 2 white or 4 black, 1 white or 5 black, 0 white) Call these events A, B, C. Ask, "What is the formula for P(AUBUC)?" The fact that these are mutually exclusive should be stressed and therefore:

$$P(AUBUC) = P(A) + P(B) + P(C) = \frac{{}_5C_3 \cdot {}_3C_2}{{}_8C_5} + \frac{{}_5C_4 \cdot {}_3C_1}{{}_8C_5} + \frac{{}_5C_5 \cdot {}_3C_0}{{}_8C_5}$$

   (An alternative solution could be to add the probabilities of 0 black, 5 white + 1 black, 4 white + 2 black, 3 white and subtract this sum from 1.)

2. Three cards are drawn at random from a standard deck of playing cards without replacement. What is the probability that:
   a) all 3 cards are aces?
   b) at least one of the cards is a picture card?
   c) there is at most one heart?
   d) each card has a different face value?

SUMMARY:

To compute probabilities, we could use $P(E) = \frac{n(E)}{n(S)}$. To find n(E) and n(S) we determine if order is important. If it is not, we use combinations to obtain both expressions.

HOMEWORK:

1. Find the probability that if two cards are drawn at random from an ordinary deck
   a) both are sixes $\left(\frac{{}_4C_2}{{}_{52}C_2}\right)$
   b) both are red $\left(\frac{{}_{26}C_2}{{}_{52}C_2}\right)$
   c) one is red, the other black $\left(\frac{{}_{26}C_1 \cdot {}_{26}C_1}{{}_{52}C_2}\right)$
   d) both are picture cards $\left(\frac{{}_{12}C_2}{{}_{52}C_2}\right)$
2. An urn contains 3 red, 1 blue and 2 green marbles. Two are selected at random. Find the probability
   a) both are red $\left(\frac{{}_3C_2}{{}_6C_2}\right)$
   b) one is red, the other green $\left(\frac{{}_3C_1 \cdot {}_2C_1}{{}_6C_2}\right)$
   c) both are blue (0)
3. From a group of 6 men and 3 women a committee of four individuals is selected. What is the probability that the committee:
   a) includes 2 women and 2 men?
   b) includes at most one woman?
   c) includes at least one woman?
   d) includes at least one man?

Lesson 54

AIM: How can we use combinations to solve more challenging probability and counting problems?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Distinguish between a counting and a probability problem.
2. Distinguish between problems requiring permutations or combinations.
3. Apply the permutation and combination formulas to solve problems involving committees, coins, words, numerals and matching of elements from an urn.
4. Determine the probability of being dealt different poker hands.

VOCABULARY: no new terms

MOTIVATION:

Say, "From this class, a committee of two is to be formed consisting of one boy and one girl. How many ways may this be done?" If, for instance, there are 20 boys and 14 girls then the counting principle gives 20 x 14 = 280 ways. This is a counting problem.
Now ask, "What is the probability that ________ and ________ (use names from class) are on the committee?" $(\frac{1}{280})$ This is a probability problem.

DEVELOPMENT:

1. Two letter "words" are formed from the letters in the set a,b,c,d. Ask, "How many ways may this be done (no letter is to be repeated)?" The counting principle or ${}_4P_2$ yields 4(3) = 12 ways if we consider order. What is the probability that "a" is one of the letters used? This can happen by having the "a" in the first or second

position. We can have "a" in the first position in 1x3 = 3 ways and in the second position in 3x1 = 3 ways. Therefore $P(E) = \frac{3+3}{12} = \frac{1}{2}$.
Some students may suggest that the above is really a combination problem since we are concerned with how many ways two letters can be selected (${}_4C_2 = 6$) and then, since one letter is to be "a" how many ways the other letter can be selected (${}_3C_1 = 3$)(order is unimportant). The result is of course the same $\frac{3}{6} = \frac{1}{2}$.

2. Ask, "What is the probability of drawing a flush (all cards of the same suit) in a five card poker hand?" Here we could consider first the probability of drawing a particular suit, for instance, all hearts, $\frac{{}_{13}C_5}{{}_{52}C_5}$, and then multiply by 4 since we must also consider all clubs, spades and diamonds. This is a combination since order is unimportant.

APPLICATIONS:

1. Mrs. Fox has 2 quarters and 6 dimes in her pocket. She takes the three coins without looking. What is the probability the coins total:
   a) 30¢ $(\frac{{}_6C_3}{{}_8C_3})$
   b) 45¢ $(\frac{{}_2C_1 \cdot {}_6C_2}{{}_8C_3})$
   c) 60¢ $(\frac{{}_2C_2 \cdot {}_6C_1}{{}_8C_3})$
   d) 50¢ (0)
   e) $\geq$ 30¢ (1)
2. The letters in the word computer are rearranged at random. What is the probability that:
   a) The first letter is a vowel?
   b) The first and last letters are vowels?
   c) The vowels appear consecutively?
   d) If a subset of 5 letters is chosen at random, one of the letters is a vowel?

3. On an examination, a student is to select any 4 problems out of 7. If John chooses the problems randomly, what is the probability that John has selected problem 7? That he has selected 2 even numbered problems and 2 odd numbered problems?

SUMMARY:

To compute probabilities, we use $P(E) = \frac{n(E)}{n(S)}$. If order is to be considered, we use permutations or the fundamental counting principle. If order is unimportant, we use combinations to find n(E) and n(S).

HOMEWORK:

1. A committee is to be chosen from a set of 7 women and 4 men. How many ways are there to form the committee if:
   a) The committee has 4 women and 2 men. $({}_7C_4 \cdot {}_4C_2)$
   b) The committee can be of any size but must have as many men as women.
   $({}_7C_1 \cdot {}_4C_1 + {}_7C_2 + {}_4C_2 \cdot {}_7C_3 + {}_4C_3 + {}_7C_4 \cdot {}_4C_4)$
   c) The committee has 5 people, one of whom must be Mr. Glatzer. $({}_{10}C_4)$
   d) If 5 are selected at random, what is the probability that Mr. Glatzer is on the committee?
   $\left(\frac{{}_{10}C_4}{{}_{11}C_5}\right)$
2. What is the probability that a seven card poker hand contains 4 of a kind?

   $$\frac{13 \cdot {}_4C_4 \cdot {}_{48}C_3}{{}_{52}C_7}$$
3. What is the probability that a 6 digit license plate (repetition allowed) ends with 212? $\frac{10^3}{10^6}$
4. The letters in the word BINARY are arranged at random. What is the probability that:
   a) The word begins and ends with a consonant?
   b) The consonants appear consecutively?
   c) If a subset of 4 letters is chosen, at least one of the letters is a vowel?

Lesson 55

AIM: What is meant by identity, inverse and closure for the operations of '+' and 'x'?

PERFORMANCE OBJECTIVES: The student will be able to...

1. State the identity element for '+'(0) and 'x'(1).
2. State the inverse element for '+'(-a) and for 'x'$\frac{1}{a}$, a $\neq 0$.
3. State why different sets of numbers (i.e., positive integers, evens, etc.) are or are not closed under different operations.

VOCABULARY: additive identity, additive inverse, multiplicative identity, multiplicative inverse, closure, operation

MOTIVATION:

Ask, "What number, when multiplied by 8 gives 8, or when multiplied by 5 gives 5, etc."
This number is called the identity for multiplication. (A discussion of the word "identity in non-mathematical contexts" might be useful to clinch the meaning.) In general $a \cdot 1 = a$ for any value of a.
The discussion could be extended here to include the identities for division and subtraction (1, 0).

DEVELOPMENT:

Suppose we start with the number 6. If we multiply by 4, we obtain $6(4) = 24$, suppose, after having done this, we wish to multiply 24 by a number to obtain our original number, 6. Ask, "What number must we multiply by? $(\frac{1}{4})$ What relationship does $(\frac{1}{4})$ have to 4?" (They are reciprocals.) Note $6(4)(\frac{1}{4}) = 6(1) = 6$. $(\frac{1}{4})$ is called the multiplicative inverse of 4 since their product $4 \times (\frac{1}{4}) = 1$ which, as we recall is the identity for multiplication.

Find the multiplicative inverse of -3. ($-\frac{1}{3}$, since $-3\cdot-\frac{1}{3} = 1$)
$\frac{2}{3}$? ($\frac{3}{2}$), $-\frac{2}{9}$? ($-\frac{9}{2}$), a ? ($\frac{1}{a}$, $a \neq 0$), $\frac{a}{b}$? ($\frac{b}{a}$)
0 ? (does not exist)

Now suppose we again start with 6, but this time add 4 to obtain 6 + 4 = 10. Ask, "What number can we now 'add' to 10 to obtain our original 6? (-4) What relationship does 4 have to -4?" (They are opposites.) Note 6 + 4 + (-4) = 6 + 0 = 6. -4 is called the additive inverse of 4 since their sum, 4 + (-4) = 0, which as we recall is the identity for addition.

Find the additive inverse of -2(2), $\frac{2}{3}$? ($-\frac{2}{3}$), $-\frac{4}{5}$? ($\frac{4}{5}$), a ? (-a). Ask, "What number is its own additive inverse? (0) Its own multiplicative inverse?" (1) Note the inverses change according to the number we are considering, whereas the identities of +, · are always 0, 1.

Consider the set $A=\{-1,0,1\}$. If we multiply any 2 elements of this set, the answer is also a member of this set. We refer to this property as closure. An operation on a set of numbers is said to be closed under this operation if the result of the operation is a unique number that is a member of that set. Is set A closed under the operation of division? subtraction? addition?

The problems 4 + 8, 4(8), 6 + 10, 2(12) - in other words, adding or multiplying even integers, yield an even integer. Since the product or sum of two even integers is always a unique even integer, we say that for multiplication or addition, the even integers are closed, i.e., the product or sum is another even integer.

Ask, "Are the even integers closed for division? (No, $\frac{4}{8} = \frac{1}{2}$ is not an even integer.) ...subtraction?" (Yes - if we include 0 and the negative evens.)

Consider this table where could be multiplication, addition or some other operation - even squaring and then subtracting 5. Let a,b,c,d denote some numbers.

△	a	b	c	d
a	b	d	a	c
b	d	c	b	a
c	a	b	c	d
d	c	a	d	b

What is (a) a △ b? (d)

(b) (b △ c) △ d? (a)

What element, if any, is the identity element for this operation? (c)

What element, if any, is the inverse of a? (d) of d? (a)

Is this system closed under the operation of △ ? (yes)

APPLICATION:

1. Under which arithmetic operations, if any, are the following sets of numbers closed?
   a. A = {0,1,2,3,4...}
   b. B = {0,1/2, 1,2}
   c. C = {positive and negative fractions}
   d. D = {positive odd integers}
2. The table below represents the operation ▱ for the set c,d,e,f .

▱	c	d	e	f
c	c	d	e	f
d	d	e	f	c
e	e	f	c	d
f	f	c	d	e

   a. What is the identity element for the system? (c)
   b. What is the inverse of f? (d)
   c. If element h had been a member of this table, what property would the set {c,d,e,f} not possess? (closure)

SUMMARY:

For the set a,b,c,d and the operation $\oplus$, if x represents the identity element and y represents the inverse of the element c, then which of the following pairs of statements are true?

(1) $c \oplus x = c$
$c \oplus y = x$

(3) $c \oplus x = c$
$c \oplus y = x$

(2) $c \oplus x = x$
$c \oplus y = c$

(4) $c \oplus x = y$
$c \oplus y = x$

HOMEWORK:

1. Find the identity, x.
   - a) $5x = 5(1)$
   - b) $6 + x = 6(0)$
   - c) $-\frac{2}{3}x = -\frac{2}{3}(1)$
   - d) $x - 4 = -4(0)$
2. Find the inverse, y.
   - a) $-\frac{2}{3}y = 1(-\frac{3}{2})$
   - b) $4y = 1(\frac{1}{4})$
   - c) $y - \frac{2}{3} = 0(\frac{2}{3})$
   - d) $-5 + y = 0(5)$
3. Explain why or why not the following are closed with the set indicated.
   - a) multiplication of odd integers (closed)
   - b) addition of odd integers (not closed)
   - c) squaring of odd integers (closed)
   - d) doubling and adding one to the integers (closed)
4.

*	a	b	c	d
a	b	d	a	c
b	d	c	b	a
c	a	b	c	d
d	c	a	d	b

   - a) What is a * b? (d)
   - b) What is (b * c) * d? (a)
   - c) What element, if any, is the identity element for this operation?
   - d) What element, if any, is the inverse of a? (d) of d? (a). Is this system closed under the operation of * ? (yes)

Lesson: 56

AIM: What is meant by the commutative, associative and distributive properties of a mathematical system?

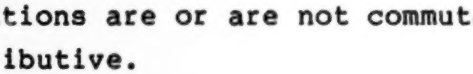
PERFORMANCE OBJECTIVES: The student will be able to...

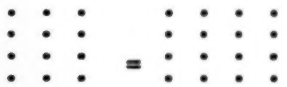
1. State why different sets of numbers under different operations are or are not commutative, associative or distributive.
2. Determine from an operation defined by a table
   a) the identity element;
   b) whether each element has an inverse;
   c) if there is closure, associativity, commutativity.

VOCABULARY: commutative, associative, distributive mathematical system.

MOTIVATION:

Ask, "Why is 4 + 6 = 6 + 4 and 4(3) = 3(4)?" Show with points in a plane, the geometric interpretation of the above, i.e.,





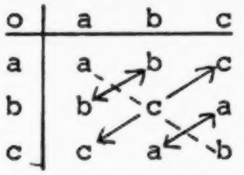
In general, $a + b = b + a$ and $ab = ba$ are known as the commutative laws for addition and multiplication. Are there such laws for division or subtraction? (No, $\frac{a}{b} \neq \frac{b}{a}$, $a - b \neq b - a$.)
Similarly, demonstrate intuitively the justification for the associative and distributive laws. Using the terms "commute", "associate", and "distribute" in non-mathematical contexts may help students remember each name. In general, $a(bc) = (ab)c$ and $(a + b) + c = a + (b + c)$ and $a(b + c) = ab + ac$. (Note there are two operations here.)

DEVELOPMENT:

Now consider the following table:

°	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Ask, "Is this set commutative? (Yes, since a°b = b°a, b°c = c°b, a°c = c°a.) ...associative? (Yes, (a°b)°c = a°(b°c).) Point out that a table is "commutative" if the elements are positioned symmetrically with respect to the diagonal



o	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Can we consider the distributive law here?" (No, since there is only one operation given here.)

The above table, where each letter may represent a number, or, as we shall see later, the movement of objects, is an example of a mathematical system. Here a,b,c are the "elements" and ° is the operation. We investigate if the above laws are satisfied and if there is closure, an identity and inverse. Systems like this are useful in other subjects besides mathematics, notably physics.

Consider,

#	a	b	c
a	b	a	c
b	c	a	b
c	a	c	b

where # denotes the operation. Ask, "Is this system closed? Why? (There are unique results, a,b,c under the operation.) What must first be determined before we check if each element has an inverse?" (What the identity is.) The class should find there is no identity and therefore no inverses are possible. Further, the associative and commutative laws do not hold.

APPLICATIONS:

1.

⊙	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Have the class find the identity (1). Ask, "Now, if $2^{\odot}x = 1$, what do we call x?" Solve for x using the table (x = 3).

Have the class check that the associative and commutative laws hold.

2. Given the set $\{w,x,y,z\}$ and the operation # as shown in the accompanying table, which statement is not true?
   1) the identity for the set is y.
   2) the set is closed under #
   3) the set is commutative under #
   4) every element of the set has an inverse

#	w	x	y	z
w	x	z	w	x
x	z	y	x	w
y	w	x	y	z
z	x	w	z	y

3. Given the set $\{a,b,c,d\}$ and the operation as shown in the accompanying table, except for the second row which has been omitted. If the operation Δ is commutative, which could be the second row?

(1) a b c d
(2) b c d a
(3) c d a b
(4) d a b c

Δ	a	b	c	d
a	b	c	d	a
b				
c	d	a	b	c
d	a	b	c	d

SUMMARY:

1. Using a,b,c, and the operation (+), write the commutative, associative and distributive laws.
2. Ask, "What else do we look for in a mathematical system?" (closure, inverse and identity)

HOMEWORK:

1. If the following is part of a commutative system, find the value of c°b. (b)

°	a	b	c
a	a	b	c
b	a	c	b
c			

2. Given the operations * and # in the tables below:

*	a	b	c	d
a	a	b	c	a
b	c	d	a	b
c	b	a	d	c
d	a	b	c	d

#	a	b	c	d
a	b	c	d	a
b	c	d	a	b
c	d	a	b	c
d	a	b	c	d

   a) Is the set closed under the operations + and #? (yes)
   b) What is the identity element for *? (d)
   c) What is the inverse of element c under the operation *? (c)
   d) Find the value of a*(b#c). (a*a = a)
   e) Find the value of (a*b)#(a*c). (c#b = a)

3. Given the elements (M,A,T,H) and the operation #, as shown in the accompanying table.
   a) What is the identity element for #?
   b) What is the inverse of M?
   c) Is the system commutative?
   d) Find x if H # x = A
   e) Which statement illustrates associativity?

      (1) H # T = T
      (2) H # H = H
      (3) M # T = T # M
      (4) (M#A) # T = M # (A#T)

#	M	A	T	H
M	A	T	H	M
A	T	H	M	A
T	H	M	A	T
H	M	A	T	H

Lesson: 57

AIM: What is clock arithmetic?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Construct clocks of 2, 4, 7, 12 elements ('+' and 'x' tables).
2. Use such clocks to solve equations such as $5 + x = 2$, $5x = 1$, $x^2 = 1$.
3. Solve problems such as finding what day of the week 53 days from today will be.
4. Interpret the modular notation $(Z_n, +, \cdot)$

VOCABULARY: Clock arithmetic and modular arithmetic. Congruent numbers is an optional term.

MOTIVATION:

Ask, "What would a clock look like if there are only two hours in a day?" In reality, there are many times when 0 and 1 are the only digits used - in the machine language of computers for example, each number is converted into the binary system where 0 and 1 represent the off and on positions of an electrical switch (similar to T and F in truth tables). Let us construct a table denoted by (Clock 2, +) where we are adding the numbers in this system.

+	0	1
0	0	1
1	1	0

Have the class fill in the table, either by referring to the clock, or imagining a car odometer with only the digits 0, 1.

DEVELOPMENT:

Questions regarding, e.g., the identity, associativity etc. may now be considered by referring to the table. Note in this table $1 + 1 = 0$. Ask, "What time would it be 15 hours from the start of a day?" (1 o'clock)

Construct the multiplication table for such a system.

°	0	1
0	0	0
1	0	1

The tables for clocks of, e.g., 2, 4, 7, 12 elements can now be considered. Refer to these as clock 2, clock 4, clock 7, and so forth. Also advise students that "clock arithmetic" is sometimes referred to as "modular arithmetic." A clock 4 system of arithmetic may also be referred to as mod 4, and so forth. For instance, using the accompanying table

°	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	3	0	1	2

the class may not only check the properties (note 2 has no inverse) within the system, but also solve such equations as $2x = 1$ ($x = 2$) or $x^2 = 1$ ($x = 1$, $x = 3$).

The last answer can then be viewed in terms of the table for addition:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Here 1 and 3 are additive inverses which is consistent with solving $x^2 = a$ ($a \geq 0$) for the real numbers.

APPLICATIONS:

1. For the above table, equations such as $x + 2 = 1$ can be solved ($x = 3$). Introduce modular notation. The set of integers is usually denoted by the letter Z. In the above tables a clock 4 (mod 4) system of arithmetic can be denoted by Z4. $(Z_4, +, .)$ represents a clock 4 arithmetic system with respect to the operations of addition and multiplication.

To find the equivalent of a large number, e.g., 45 in such a system, we must introduce the concept of congruent numbers, i.e., numbers which have the same remainder when divided by the "modulus" - in this case, 4. Since 45 = 4(11) + 1, it is equivalent to, e.g., 61 = 4(15) + 1, or 1 = 0(1) + 1. Students may discover this when asked questions like, "If it is Spring now, what season will be 45 seasons from now? Let 0 = Spring; 1 = Summer, 2 = Fall, 3 = Winter, and use the above to obtain Summer.

2. Given $(Z_3, +, 0)$ $Z_3 = (0,1,2)$ + is addition clock 3 and 0 is multiplication clock 3
   a) Construct an addition table and a multiplication table for Z_3 as defined.
   b) Which element, if any, does not have a multiplicative inverse?
   c) Solve for all values of x in the clock 3 (mod 3) system:
      1. $x + x = 2$
      2. $x \cdot x = 2$
      3. $x + x = x \cdot x$

SUMMARY:

1. Given $(Z_7 +, .)$,
   a) What are the elements of this system?
   b) Which element, if any, does not have a multiplicative inverse?

2. What clock would you construct to solve problems related to the months of the year?

3. What topic did you study in mathematics prior to high school which is related to clock arithmetic? (number bases)

HOMEWORK:

1. Construct tables for Clock 7 and Clock 12 for +.

   Use these tables to solve the following.

Clock 7	Clock 12
$x + 5 = 1$ $(x = 3)$	$x + 5 = 1$ $(x = 8)$
$x^2 = 6$ $(\emptyset)$	$x^2 = 6$ $(x = 9, x = 3)$

   Optional: using $x - a = b \longrightarrow a = b + a$, solve clock 7.
   $(x - 5 = 5)$ $(x = 3)$

   Optional: using $\frac{x}{a} = b$ $(x = ab)$, solve in clock 7.
   $\frac{x}{3} = 4$ $(x = 5)$

   Solve in clock 12: $3x = 9$. $(x = 3, x = 7, x = 11)$

2. Construct tables for clock 5 (mod 5) system of arithmetic.
   a) What is the solution set for the equation $x+4=3$?
   b) What element, if any, does not have a multiplicative inverse?

Lesson: 58

AIM: What is a group?

PERFORMANCE OBJECTIVES: The student will be able to...

1. State the properties of a group.
2. Determine if a given system is a group.
3. Use the symbol (G, *) to designate groups.

VOCABULARY: group

MOTIVATION:

In the previous assignment, the tables for clock 7, + and were constructed. Let us now consider the table for +.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

Assign rows of students to investigate closure, associativity, identity and inverses.

DEVELOPMENT:

The class should discover that the system is closed, associative (as is addition in all clocks), the identity of 0 and each number has an inverse. Such a system is called a group and denoted in general by (G, *) where G is the set we are considering (in this case, the integers 0, 1, 2, 3, 4, 5, 6) and * is binary operation (in this case +).
Have the class refer to their tables for (clock 7, •) and determine if that system can be termed a group, i.e., if it is closed and associative, having both an identity and inverses. They should determine that it is a group only if 0 is excluded since 0 does not have an inverse.

Again refer to the previous assignment, this time to the tables for Clock 12, 1 + and °. If this problem was not assigned, have them construct tables for clock 4. In either case, it should be discovered that under the operation °, each element (number) failed to have an inverse and hence the system is not a group. The class may be informed that (Clock M, °) will be a group if and only if (<——>) M is a prime number.

MEDIAL SUMMARY:

A mathematical system consisting of a set G and a binary operation *, is a group if it satisfies the following four properties.

1. G is closed under *.
2. * is associative on G.
3. There exists an identity element e in G such that: $x*e = e*x = x$, for all x in G.
4. There exists an inverse element x^{-1} in G for each element x in G such that: $x \cdot x^{-1} = x^{-1} \cdot x = e$.

Note that x^{-1} is a common notation that is read as "the inverse of element x". The "-1" in x^{-1} is not treated as an exponent in the usual sense. Although the identity and inverse elements must satisfy the commutative property, the commutative property does not necessarily hold with any pair of elements in G.

APPLICATIONS:

1. Ask, "Does the set of integers form a group under +?" Why? (Yes, since it is closed, associative, has an identity of 0 and each number x has an inverse of -x.) "Does the set of integers under ° form a group?" They are certainly closed, associative and have an identity of 1, but the inverses are not all integers.
   It may be mentioned that although groups considered above are commutative, there are non-commutative groups (see Ref. 1 for an example on permutations of chairs or Ref. 2 on translation of the plane).

2. Given set A = {m,s,t,r} and the associative operation #
as defined by the accompanying table

#	m	s	t	r
m	r	m	s	t
s	m	s	t	r
t	s	t	r	m
r	t	r	m	s

a) Verify that (s#m)#t = s#(m#t)
b) What is the identity element?
c) What is the inverse of m?
d) Does the system (a,#) form a commutative group?
e) Find x: x#x = r

SUMMARY:

1. What properties define a group?
2. If (G,*) is a group, which elements, if any, satisfy the commutative property?

HOMEWORK:

Show that the following are groups.

1) The rationals or reals under addition
2) The even integers under addition
3) The system below

°	A
A	A

Show that the following are not groups.

4) Whole numbers + (no inverses)
5) Integers (not associative)
6) The system below

°	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Lesson: 59

AIM: How can we identify groups in various mathematical systems?*

PERFORMANCE OBJECTIVES: The student will be able to...

1. Show the properties of groups apply to such problems as
   a) the permutation of three chairs
   b) the rotation of triangles or wheels
   c) demonstrating the cancellation law or $a \cdot x = b$
   $x = a^{-1} \cdot b$

(*Note: This lesson is more suitable for more advanced classes, other classes may benefit from an additional day in examining systems defined by tables.)

VOCABULARY: no new terms

MOTIVATION:

In the previous lesson, reference was made to the permutation of chairs. Label three chairs (or students) in order from left to right. Let a new arrangement of chairs be defined by

a) $\begin{matrix}1&2&3\\ \downarrow&\downarrow&\downarrow\\ 3&2&1\end{matrix}$ b) $\begin{matrix}1&2&3\\ \downarrow&\downarrow&\downarrow\\ 2&1&3\end{matrix}$ c) $\begin{matrix}1&2&3\\ \downarrow&\downarrow&\downarrow\\ 3&1&2\end{matrix}$

d) $\begin{matrix}1&2&3\\ \downarrow&\downarrow&\downarrow\\ 2&3&1\end{matrix}$ e) $\begin{matrix}1&2&3\\ \downarrow&\downarrow&\downarrow\\ 1&3&2\end{matrix}$ f) $\begin{matrix}1&2&3\\ \downarrow&\downarrow&\downarrow\\ 1&2&3\end{matrix}$

Let a*b consist of doing a first then b so that

$\begin{matrix}\text{a)} & 1&2&3\\ & \downarrow&\downarrow&\downarrow\\ & 3&2&1\\ \text{b)} & \downarrow&\downarrow&\downarrow\\ & 3&1&2\end{matrix}$ a*b is the same as $\begin{matrix}1&2&3\\ \downarrow&\downarrow&\downarrow\\ 3&1&2\end{matrix}$ which is c

This means, for instance, chair 3 goes first into position under the permutation a. Permutation b takes whatever is in position 1 (in this case chair 3) and moves it to position 2.

DEVELOPMENT:

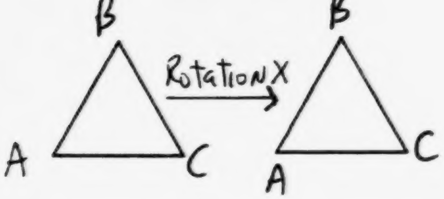
Ask the class to first determine the result of b*a and then to complete the following table.

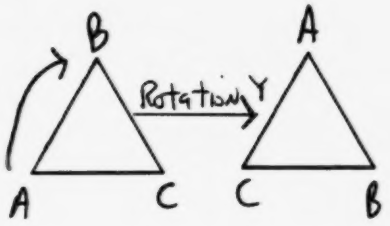
*	i	a	b	c	d	e
i						
a			c			
b		d				
c						
d						

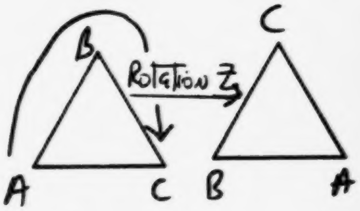
Note that b*a consists of doing

$$\begin{array}{cccc} & 1 & 2 & 3 \\ b & \downarrow & \downarrow & \downarrow \\ & 2 & 1 & 3 \\ a & \downarrow & \downarrow & \downarrow \\ & 2 & 3 & 1 \end{array} \quad \text{or} \quad \begin{array}{ccc} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{array}$$

thus yielding d. When completed, ask the class if this table defines a group. (yes) Ask, "What type?" (non commutative) Consider next the rotation of an equilateral triangle as follows.







The following table summarizes such a rotation. Have the class show this system forms a commutative group.

°	x	y	z
x	x	y	z
y	y	z	x
z	z	x	y

APPLICATIONS:

If we denote the inverse of a by a^{-1}, we can show that in a group (G,0) where G may be the seals and 0 may be addition or multiplication that $a^\circ x = b$ leads to $x = a^{-1}b$ (which is our basic algebraic technique of solving such equations as $3 + x = 5$ or $3x = 12$). Since a^{-1} exists (G is a group) then

$$\begin{aligned} a^{-1}\circ\ (a^{-1}\circ\ a)^\circ x &= a^{-1}\circ\ b \\ (a^{-1}\circ\ a)^\circ x &= a^{-1}\circ\ b \\ 1^\circ x &= a^{-1}\circ\ b \\ x &= a^{-1}\circ\ b \end{aligned}$$

Have students supply reasons - (substitution, associativity, inverse, identity).

SUMMARY:

Name a commutative group. A non-commutative group. Ask, "What are the properties of a group? Which elements in a non-commutative group must satisfy the commutative property?"

HOMEWORK:

1. Let G be the set of rotations on a wheel through angles equal to 60N where N is an integer. Find
   a) the identity (N = 0)
   b) the inverse of a rotation through 60N (-60N)
2. The following is called the Klein Four Group.

$\circ$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

   a) Why is there closure?
   b) Verify $(a \circ b) \circ c = a \circ (b \circ c)$.
   c) What is the identity? (e)
   d) What are the inverses of a, b, c? (a,b,c)

Lesson: 60

AIM: What is a field?

PERFORMANCE OBJECTIVES: The student will be able to...

1. State the properties of a field.
2. Determine if a given system is a field.
3. Use the notation $(F, +, \circ)$ to denote fields.

VOCABULARY: field

MOTIVATION:

Ask, "In defining groups, we were concerned with which four properties? (closure, associativity, identity and inverses) What other properties have we recently studied?" (commutativity and the distributive law)
For the distributive law, we needed two operations. (Recall $a(b + c) = ab + ac$ or more generally $a\#(b \circ c) = (a\#b)\circ(a\#c)$.
Now consider the system for clock 3, + and x.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

°	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Ask, "What type of group is $(Z_3, +)$? (commutative) Is $(Z_3, \circ)$ a group?" (No, since there is no inverse for °)

Note, however, if we exclude 0, $(Z_3, \circ)$ does form a group. What type of group? (commutative)

DEVELOPMENT:

Now let us consider the one law we have not yet considered, the distributive law. Does the system $(Z_3,+,\circ)$ obey that law, e.g., does $3(2 + 1) = 3(2) + 3(1)$? Show that this in fact is true and inform the class that since all of the above conditions have been met, this system is known as a field. Intuitively, a field, having two operations say +, ° has each of the following: a closure for +, °, associativity for +, °, commutativity for +, °, identities for +, °, inverses for +, ° (excluding 0) and the distributive property linking the two operations. Note in order to have a field, we must first have two groups. The notation, (F, +, °) may be used to denote a field.

It may be pointed out that the solution of equations such as $3 + x = 5$ and $3x = 12$ depend on group properties, while field properties are applied to solve equations with two operations such as $3x + 12 = 14$.

Formally define a field in terms of a group. For example, an algebraic system (F,+,°) consisting of a set of numbers F and two commutative operations + and °, that possess the following properties is called a field:

1. (F,+) is a commutative group
2. (F,°) is a commutative group if the identity element for + is excluded from F
3. For all elements a,b, and c in F,
   $a^\circ(b+c) = a^\circ b+a^\circ c$ and $(b+c)^\circ a = b^\circ a+c^\circ a$
   (distributive property)

APPLICATIONS:

Show that the integers, clock 2 also form a field, but the integers, clock 4 do not form a field (each element in clock 4, does not have an inverse). Ask, "What may we conclude from this?" (as with groups, for clock m, to form a field, m must be prime)

Ask, "Which of the following do you believe form a field under +, °?"

a) the rationals and reals (yes)
b) integers (no, since non-zero integers do not form a group under ° since there is not always an integer inverse)

SUMMARY:

Ask, "State the properties of a field. Can a field be non-commutatiave? (no) Name a system that is a field and one that is not."

HOMEWORK:

1. Given the clock 5 (mod 5) field (F, +, °) where F = $\{0,1,2,3,4\}$ and operations + and ° are defined below:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

°	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

2. A field consists of the set of real numbers of the form $a + b\sqrt{2}$ where a and b are rationals. Using any values for a, b, show that the following are satisfied.

   Closure for addition. For example, $(3 + 2\sqrt{2}) + (5 + 6\sqrt{2}) = 8 + 8\sqrt{2}$. (note a, b are in the sum and both are rational)

   Additive identity: $(3+5\sqrt{2}) + (0 + 0\sqrt{2}) = 3 + 5\sqrt{2}$

   Additive inverse: $(3 + 5\sqrt{2}) + (-3 - 5\sqrt{2}) = 0)$

   Multiplication closure:
   $(3 + 2\sqrt{2})(6\sqrt{2}) = 15 + 10\sqrt{2} + 18\sqrt{2} + 24 = 39 + 28\sqrt{2}$; again a and b in the product are both rational.

   Multiplication identity: $(3 + 2\sqrt{2})(1 + 0\sqrt{2}) = 3 + 2\sqrt{2}$

3. Let Q = rationals. Determine whether each of the following systems form a field.
   a. $(Q^-,+,\cdot)$ where the members of Q^- are less than 0.
   b. $(Q^+,+,\cdot)$ where the members of Q^+ are non-negative.
   c. $(H,+,\cdot)$ where $H=\{-1,0,1\}$